NONCONVEX OPTIMIZATION AND ITS APPLICATIONS

Abstract Convexity and Global Optimization

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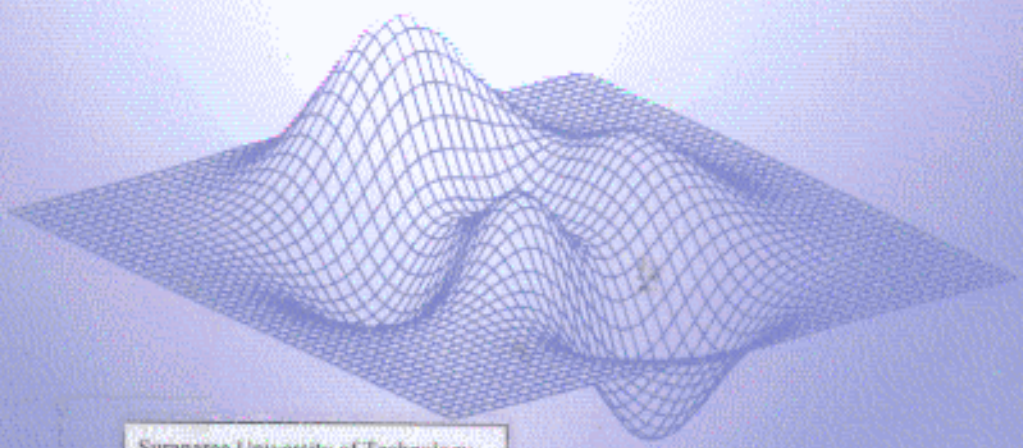

Suranaree University of Technology
31051900652012

Kluwer Academic Publishers

Contents

Preface xi
Acknowledgment xvii

1. AN INTRODUCTION TO ABSTRACT CONVEXITY 1
   1.1 Overview 1
   1.2 Preliminaries 2
   1.3 Abstract convex functions and sets 3
   1.4 Subdifferentiability 9
   1.5 Conjugation 12
   1.6 Abstract concave functions and infima of abstract convex functions 13

2. ELEMENTS OF MONOTONIC ANALYSIS: IPH FUNCTIONS AND NORMAL SETS 15
   2.1 Introduction 15
   2.2 Increasing positively homogeneous functions defined on positive orthant 18
      2.2.1 Overview 18
      2.2.2 Preliminaries 18
      2.2.3 IPH functions 19
      2.2.4 Min-type functions and IPH functions 23
      2.2.5 Abstract convexity with respect to the set of min-type functions 26
      2.2.6 Level sets of IPH functions 28
      2.2.7 Polarity for normal sets and IPH functions 30
      2.2.8 Support sets 33
      2.2.9 Subdifferential 36
      2.2.10 Concavity of the polar function 37
      2.2.11 Comparison with convex analysis 39
   2.3 Increasing positively homogeneous functions defined on the non-negative orthant 42
      2.3.1 Overview 42
      2.3.2 Preliminaries 43

2.3.3 L-convex function 44
2.3.4 Support sets 46
2.3.5 Two kinds of normality 47
2.3.6 Properties of the support sets 50
2.3.7 Subdifferentials of IPH functions 54
2.3.8 Abstract concavity and superdifferentials 57
2.4 Best approximation by normal sets 60
2.4.1 Overview 60
2.4.2 Preliminaries 61
2.4.3 Distance to a normal set 63
2.4.4 Separation 66
2.4.5 Distance to the union and the intersection of normal sets 71

3. ELEMENTS OF MONOTONIC ANALYSIS: MONOTONIC FUNCTIONS 75
3.1 Introduction 75
3.2 Increasing co-radiant functions 77
3.2.1 Overview 77
3.2.2 Definition and properties of ICR functions 77
3.2.3 ICR functions and IPH functions 80
3.2.4 Abstract convexity of ICR functions 81
3.3 Increasing convex-along-rays functions 82
3.3.1 Overview 82
3.3.2 ICAR functions: definition, examples and some properties 83
3.3.3 ICAR functions as abstract convex functions 87
3.3.4 Subdifferentiability of ICAR functions 92
3.3.5 Subdifferentiability of strictly ICAR functions 93
3.3.6 Lipschitz function and ICAR functions 95
3.4 Decreasing functions 98
3.4.1 Overview 98
3.4.2 Decreasing functions and IPH functions 98
3.4.3 Multiplicative inf-convolution 107

4. APPLICATION TO GLOBAL OPTIMIZATION: LAGRANGE AND PENALTY FUNCTIONS 113
4.1 Introduction 113
4.2 Extended Lagrange and penalty functions 117
4.2.1 Overview 117
4.2.2 Preliminaries 117
4.2.3 Extended Lagrange functions 119
4.2.4 Extended penalty functions 123
4.2.5 Examples 124
4.2.6 Support set of the dual function 125
4.2.7 Another approach 127
4.3 Extended penalization for problems with one constraint 128
4.3.1 Overview 128
4.3.2 Preliminaries 129

4.3.3 Perturbation functions 131
4.3.4 Exact penalization 138
4.3.5 Penalization by IPH functions p_k 144

5. ELEMENTS OF STAR-SHAPED ANALYSIS 153
5.1 Introduction 153
5.2 Radiant and co-radiant sets and their gauges 155
5.2.1 Overview 155
5.2.2 Radiant and co-radiant sets 155
5.2.3 Radiative and co-radiative sets 161
5.2.4 Radiant sets with Lipschitz continuous Minkowski gauges 165
5.3 Star-shaped sets and co-star-shaped sets 170
5.3.1 Overview 170
5.3.2 Star-shaped sets and their kernels 170
5.3.3 Sum of star-shaped sets and sum of co-star-shaped sets 175
5.4 Separation 180
5.4.1 Overview 180
5.4.2 Cone-separation and separation by a finite collection of linear functions 181
5.4.3 Separation of star-shaped sets 186
5.4.4 Separation of co-star-shaped sets 191
5.5 Abstract convexity with respect to general min-type functions 202
5.5.1 Overview 202
5.5.2 Positively homogeneous abstract convex functions 203
5.5.3 $\mathcal{H}$-convex functions 211
5.5.4 Subdifferentials of $\mathcal{H}_{n+1}$-convex functions 220
5.5.5 Abstract convex sets 223
5.5.6 Other classes of abstract convex functions 226

6. SUPREMAL GENERATORS AND THEIR APPLICATIONS 229
6.1 Introduction 229
6.2 Continuous and lower semicontinuous functions 231
6.2.1 Overview 231
6.2.2 Lower semicontinuous functions 231
6.2.3 Examples 237
6.2.4 ICAR extensions of functions defined on the unit simplex 239
6.3 Supremal generators for spaces of homogeneous functions 240
6.3.1 Overview 240
6.3.2 Preliminaries 241
6.3.3 Homogeneous functions of degree one 242
6.3.4 Symmetric positively homogeneous functions of degree two 244
6.4 Some applications of supremal generators 247

6.4.1 Overview 247
6.4.2 Convergence of sequences of positive functionals 248
6.4.3 The supremal rank of a compact set 251
6.5 Application to Hadamard-type inequalities 254
6.5.1 Overview 254
6.5.2 Hadamard-type inequalities for convex functions 255
6.5.3 Quasiconvex functions and P-functions 256
6.5.4 Inequalities of Hadamard type for P-functions and quasiconvex functions 262
6.5.5 Inequality of Hadamard type for ICAR functions 265

7. FURTHER ABSTRACT CONVEXITY 271
7.1 Introduction 271
7.2 Abstract Convexity with respect on a subset 272
7.2.1 Overview 272
7.2.2 Basic definitions and properties 273
7.2.3 Subdifferentiability 278
7.2.4 Conjugation and approximate subdifferentials 282
7.2.5 Abstract convexity and global minimization 288
7.2.6 Minimax result for abstract convex functions 290
7.2.7 Positively homogeneous functions 291
7.2.8 Positively homogeneous extension 294
7.2.9 Polarity for functions and sets, which are abstract convex with respect to a conic set of positively homogeneous functions 297
7.3 Some classes of abstract convex functions 304
7.3.1 Overview 304
7.3.2 Linear functions - generating sublinearity 304
7.3.3 Affine functions - generating convexity 308
7.3.4 Min-type functions - generating convexity-along-rays 315
7.3.5 Two-step functions - generating quasiconvexity. Abstract convex functions 316
7.3.6 Two-step functions - generating quasiconvexity. Abstract convex sets 321
7.3.7 Two-step functions - generating quasiconvexity. Abstract subdifferentials 329
7.3.8 Infima of families of abstract convex functions 333
7.4 Minkowski duality, c^2-lattices and semilinear lattices 335
7.4.1 Overview 335
7.4.2 The Minkowski duality 336
7.4.3 Minkowski duality for c^2-lattices and semilinear lattices 338

8. APPLICATION TO GLOBAL OPTIMIZATION: DUALITY 345
8.1 Introduction 345
8.2 General solvability theorems 347
8.2.1 Overview 347

8.2.2 Solvability theorems for systems of abstract convex functions 348
8.2.3 Further solvability results 350
8.2.4 Sublinear inequality systems 354
8.2.5 Convex inequality systems 355
8.2.6 Positively homogeneous systems 356
8.2.7 Twice continuously differentiable systems 359
8.2.8 Quasiconvex inequality systems 360
8.2.9 Systems involving functions expressible as the infima of families of convex functions 361
8.2.10 Convex maximization 362
8.3 Maximal elements of support sets and Toland-Singer formula 363
8.3.1 Overview 363
8.3.2 Maximal elements 364
8.3.3 Existence of maximal elements of support sets 367
8.3.4 Positively homogeneous elementary functions 371
8.3.5 Maximal elements of support sets with respect to conic sets of positively homogeneous functions 373
8.3.6 The Toland-Singer formula and maximal elements of support sets 375
8.3.7 Excess functions 378
8.4 Optimization of the difference of abstract convex functions 380
8.4.1 Overview 380
8.4.2 Minimization of the difference of coercive convex functions 382
8.4.3 Minimization of the difference of coercive convex and sublinear functions 386
8.4.4 Dual optimality conditions for the difference of ICAR functions 391
8.4.5 Necessary and sufficient conditions for the minimum of the difference of coercive strictly ICAR functions 393
8.4.6 Minimization of the quotient of convex functions 396

9. APPLICATION TO GLOBAL OPTIMIZATION: NUMERICAL METHODS 399
9.1 Introduction 399
9.2 Conceptual schemes of numerical methods 401
9.2.1 Overview 401
9.2.2 Special minorates of abstract convex functions 401
9.2.3 Generalized cutting plane method 404
9.2.4 Branch-and-bound methods 410
9.2.5 Tabu search 413
9.2.6 External centres method 415
9.2.7 Lipschitz programming via abstract convexity 416
9.3 Cutting angle method 418
9.3.1 Overview 418

9.3.2 Cutting angle method for ICAR functions 419
9.3.3 The subproblem 421
9.3.4 Numerical results - ICAR objective functions 424
9.3.5 Cutting angle method for increasing co-radiant functions 427
9.3.6 Numerical results - ICR objective functions 429
9.3.7 Cutting angle method for Lipschitz functions 432
9.3.8 Numerical results - Lipschitz functions 436
9.3.9 Branch-and-bound method for Lipschitz functions 438
9.4 Cutting angle method (continuation) 439
9.4.1 Overview 439
9.4.2 Cutting angle method for the minimization of IPH functions over the unit simplex 440
9.4.3 The subproblem: local minima 442
9.4.4 The subproblem: global minima 447
9.4.5 Results of numerical experiments 450
9.4.6 An exact method for solving the subproblem 454
9.5 Monotone optimization 460
9.5.1 Overview 460
9.5.2 Preliminaries 461
9.5.3 Problems of monotonic optimization 463
9.5.4 Basic properties 464
9.5.5 Proposed solution method 466
9.5.6 Convergence 468
9.5.7 Computational experience 469
References 471

Index 489