

APPLICATIONS
OF MATHEMATICS

STOCHASTIC
MODELLING
AND APPLIED
PROBABILITY

45

J. Michael Steele

Stochastic Calculus and Financial Applications



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Contents

Preface	v
1. Random Walk and First Step Analysis	1
1.1. First Step Analysis	1
1.2. Time and Infinity	2
1.3. Tossing an Unfair Coin	5
1.4. Numerical Calculation and Intuition	7
1.5. First Steps with Generating Functions	7
1.6. Exercises	9
2. First Martingale Steps	11
2.1. Classic Examples	11
2.2. New Martingales from Old	13
2.3. Revisiting the Old Ruins	15
2.4. Submartingales	17
2.5. Doob's Inequalities	19
2.6. Martingale Convergence	22
2.7. Exercises	26
3. Brownian Motion	29
3.1. Covariances and Characteristic Functions	30
3.2. Visions of a Series Approximation	33
3.3. Two Wavelets	35
3.4. Wavelet Representation of Brownian Motion	36
3.5. Scaling and Inverting Brownian Motion	40
3.6. Exercises	41
4. Martingales: The Next Steps	43
4.1. Foundation Stones	43
4.2. Conditional Expectations	44
4.3. Uniform Integrability	47
4.4. Martingales in Continuous Time	50
4.5. Classic Brownian Motion Martingales	55
4.6. Exercises	58

5. Richness of Paths	61
5.1. Quantitative Smoothness	61
5.2. Not Too Smooth	63
5.3. Two Reflection Principles	66
5.4. The Invariance Principle and Donsker's Theorem	70
5.5. Random Walks Inside Brownian Motion	72
5.6. Exercises	77
6. Itô Integration	79
6.1. Definition of the Itô Integral: First Two Steps	79
6.2. Third Step: Itô's Integral as a Process	82
6.3. The Integral Sign: Benefits and Costs	85
6.4. An Explicit Calculation	85
6.5. Pathwise Interpretation of Itô Integrals	87
6.6. Approximation in $\mathcal{H}^2$	90
6.7. Exercises	93
7. Localization and Itô's Integral	95
7.1. Itô's Integral on $\mathcal{L}_{\text{Loc}}^2$	95
7.2. An Intuitive Representation	99
7.3. Why Just $\mathcal{L}_{\text{Loc}}^2$?	102
7.4. Local Martingales and Honest Ones	103
7.5. Alternative Fields and Changes of Time	106
7.6. Exercises	109
8. Itô's Formula	111
8.1. Analysis and Synthesis	111
8.2. First Consequences and Enhancements	115
8.3. Vector Extension and Harmonic Functions	120
8.4. Functions of Processes	123
8.5. The General Itô Formula	126
8.6. Quadratic Variation	128
8.7. Exercises	134
9. Stochastic Differential Equations	137
9.1. Matching Itô's Coefficients	137
9.2. Ornstein-Uhlenbeck Processes	138
9.3. Matching Product Process Coefficients	139
9.4. Existence and Uniqueness Theorems	142
9.5. Systems of SDEs	148
9.6. Exercises	149
10. Arbitrage and SDEs	153
10.1. Replication and Three Examples of Arbitrage	153
10.2. The Black-Scholes Model	156
10.3. The Black-Scholes Formula	158
10.4. Two Original Derivations	160
10.5. The Perplexing Power of a Formula	165
10.6. Exercises	167

11. The Diffusion Equation	169
11.1. The Diffusion of Mice	169
11.2. Solutions of the Diffusion Equation	172
11.3. Uniqueness of Solutions	178
11.4. How to Solve the Black-Scholes PDE	182
11.5. Uniqueness and the Black-Scholes PDE	187
11.6. Exercises	189
12. Representation Theorems	191
12.1. Stochastic Integral Representation Theorem	191
12.2. The Martingale Representation Theorem	196
12.3. Continuity of Conditional Expectations	201
12.4. Lévy's Representation Theorem	203
12.5. Two Consequences of Lévy's Representation	204
12.6. Bedrock Approximation Techniques	206
12.7. Exercises	211
13. Girsanov Theory	213
13.1. Importance Sampling	213
13.2. Tilting a Process	215
13.3. Simplest Girsanov Theorem	218
13.4. Creation of Martingales	221
13.5. Shifting the General Drift	222
13.6. Exponential Martingales and Novikov's Condition	225
13.7. Exercises	229
14. Arbitrage and Martingales	233
14.1. Reexamination of the Binomial Arbitrage	233
14.2. The Valuation Formula in Continuous Time	235
14.3. The Black-Scholes Formula via Martingales	241
14.4. American Options	244
14.5. Self-Financing and Self-Doubt	246
14.6. Admissible Strategies and Completeness	252
14.7. Perspective on Theory and Practice	257
14.8. Exercises	259
15. The Feynman-Kac Connection	263
15.1. First Links	263
15.2. The Feynman-Kac Connection for Brownian Motion	265
15.3. Lévy's Arcsin Law	267
15.4. The Feynman-Kac Connection for Diffusions	270
15.5. Feynman-Kac and the Black-Scholes PDEs	271
15.6. Exercises	274
Appendix I. Mathematical Tools	277
Appendix II. Comments and Credits	285
Bibliography	293
Index	297