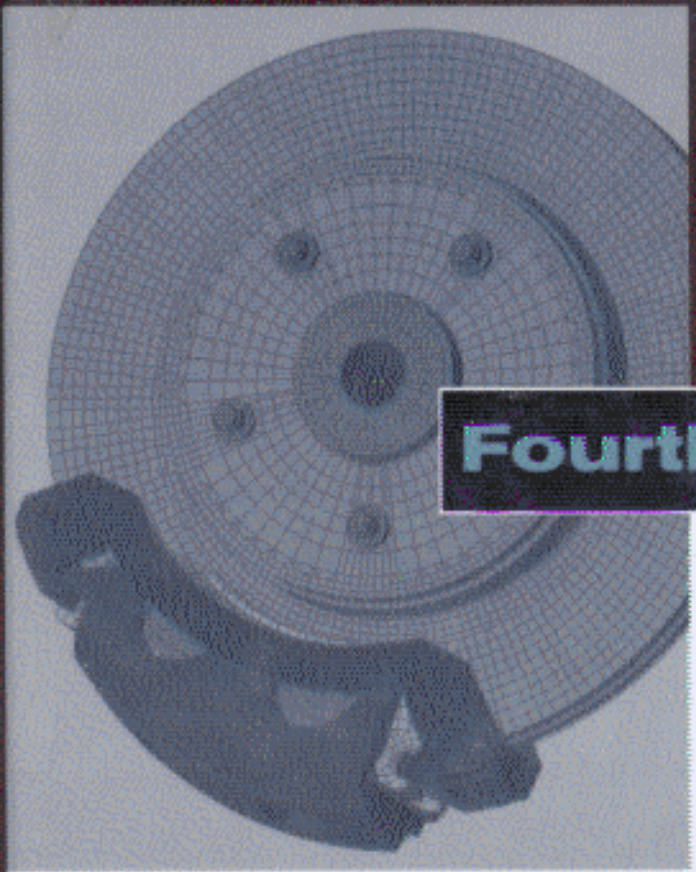
# THE FINITE ELEMENT METHOD FOR ENGINEERS



Fourth Edition

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