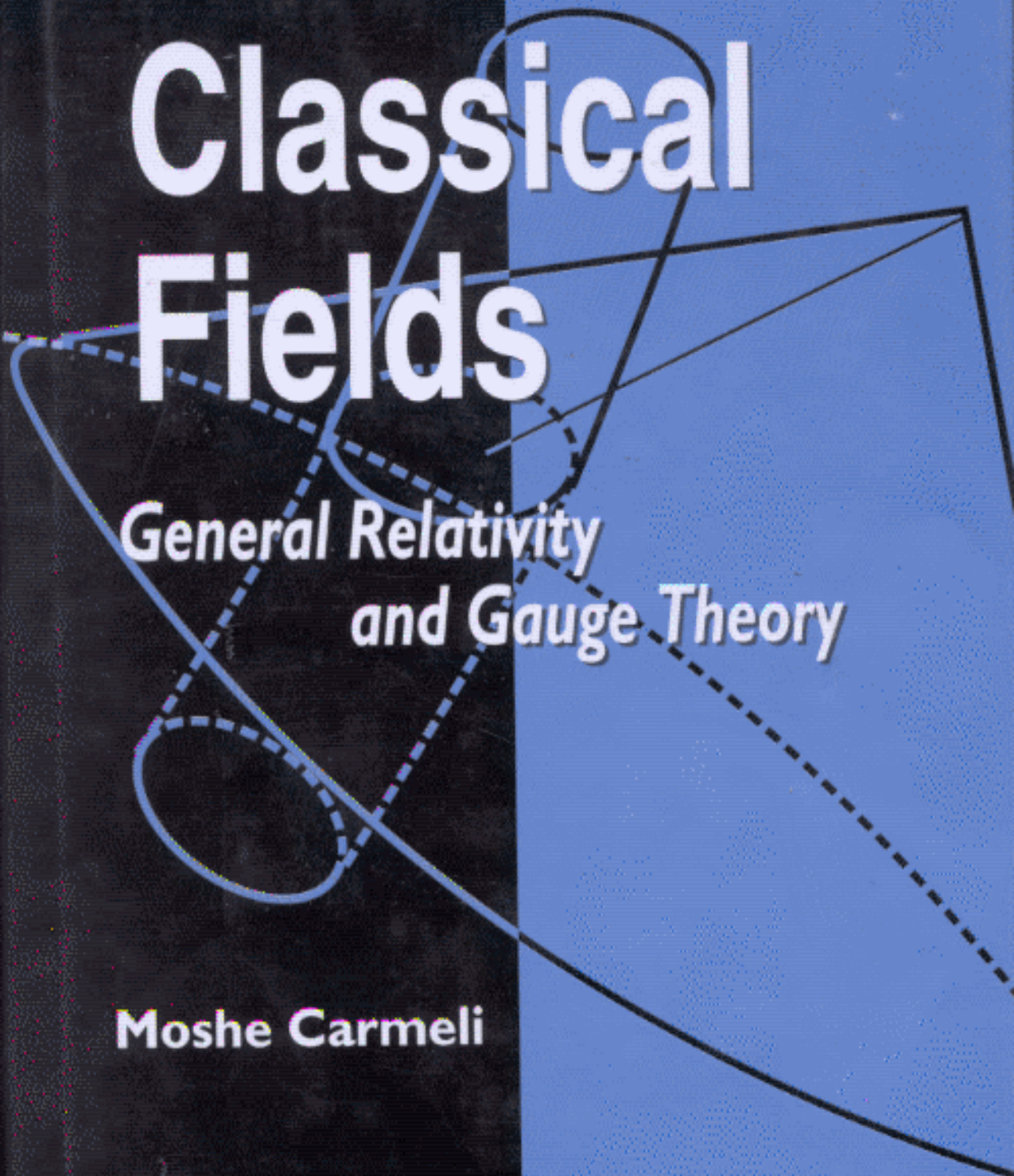
Classical Fields

General Relativity and Gauge Theory



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World Scientific

CONTENTS

1 The Gravitational Field **1**

1.1 Newtonian Gravitation 1

The Galilean group. Newtonian mechanics. Newton's theory of gravitation. Problems.

1.2 Basic Properties of the Gravitational Field 10

1.3 Null Experiments 13

1.4 Principle of Equivalence 16

1.5 Principle of General Covariance 17

Suggested References.

2 The Geometry of Curved Spacetime **20**

2.1 Transformation of Coordinates 20

Contravariant vectors. Invariants. Covariant vectors.

2.2 Tensors 23

Definition of a tensor. Tensor algebra.

2.3 Symmetry of Tensors 26

Problems.

2.4 The Metric Tensor 33

2.5 Tensor Densities 35

Definition of a tensor density. Levi-Civita tensor densities. Problems.

2.6 The Christoffel Symbols 45

Transformation laws for Christoffel symbols. Some useful formulas. Geodesic coordinate system. Problems.

2.7 Covariant Differentiation 51

Rules for covariant differentiation. Some useful formulas. Problems.

2.8 Geodesics 61

Affine parameter. Null geodesics. Problems.

2.9 The Riemann Curvature Tensor 67
The Ricci identity. Symmetry of the Riemann curvature tensor. Ricci tensor and scalar; Einstein tensor. The Weyl conformal tensor. Properties of the Weyl conformal tensor. Problems.
2.10 Differential Identities 80
The Bianchi identities. The contracted Bianchi identities. Problems. Suggested References.

3 The Einstein Field Equations 84

3.1 The Gravitational Field Equations 84
Derivation of the gravitational field equations. Properties of the Einstein field equations.
3.2 The Newtonian Limit of the Einstein Field Equations 88
Problems.
3.3 Action Integral for the Gravitational Field 93
Problems.
3.4 Equations of Electrodynamics in the Presence of Gravitation 105
Problems.
3.5 Lie Derivative 113
Problems.
3.6 Structure of the Spacetime 122
The Killing equation. Simple example: the Poincaré group. Problems.
3.7 Stationary and Static Gravitational Fields 130
3.8 Tetrad Formulation of the Einstein Field Equations: The Newman–Penrose Equations 135
The null tetrad. The spin coefficients. Tetrad components. The Newman–Penrose equations. The optical scalars. The electromagnetic field.
3.9 Perturbation on Gravitational Background 143
Decoupled gravitational equations. Decoupled electromagnetic equations. Problems.
3.10 Coordinate Conditions 151
Definition of coordinate conditions. deDonder coordinate condition, harmonic coordinate system. Problems.
3.11 Initial-Value Problem 152
Problems. Suggested References.

4 Gravitational Fields of Elementary Mass Systems 155

4.1 The Schwarzschild Metric 155
Problems.

4.2 The Kruskal Coordinates 163

The Eddington–Finkelstein form for the spherically symmetric metric. Maximal extension of the Schwarzschild metric.

4.3 Gravitational Field of a Spherically Symmetric Charged Body 168

4.4 Gravitational Field with Rotational Symmetry 172

Problems.

4.5 Field of Particle with Quadrupole Moment 177

Problems.

4.6 The Vaidya Radiating Metric 183

Derivation. The Vaidya metric in null coordinates. Problems.

4.7 The Tolman Metric 189

Fluid without pressure. Comoving coordinates. Field equations. Solutions of the field equations. Problems.

4.8 The Einstein–Rosen Metric 198

Cylindrical gravitational waves. Periodic solutions. Pulse solutions. Suggested References.

5 Properties of the Gravitational Field 205

5.1 Weak Gravitational Field 205

Linear approximation. The linearized Einstein equations. Problems.

5.2 Gravitational Red Shift 213

5.3 Motion in a Centrally Symmetric Gravitational Field 215

5.4 Deflection of Light in a Gravitational Field 222

Problems.

5.5 Other Tests of General Relativity Theory 227

Detection of gravitational waves. Delay of radar pulses in gravitational field. Problems.

5.6 Gravitational Radiation 234

The light cone at infinity. The geometry of the manifold $\mathfrak{N}$. The general relativistic case. Gravitational waves. Helicity and polarization of gravitational waves. Choice of coordinate system—Bondi coordinates. Problems.

5.7 The Energy-Momentum Pseudotensor 247

Conservation laws in the presence of gravitation. Energy-momentum pseudotensor. Four-momentum. Angular momentum. Gravitational radiation from isolated system. The quadrupole radiation formula. Energy loss by two bodies. Problems.

5.8 Gravitational Bremsstrahlung 255

Spectral resolution of intensity of dipole and quadrupole. Radiation of low frequencies in collision. Gravitational radiation in nonrelativistic

collisions. Solar gravitational radiation. Total gravitational radiation. Comparison with classical sources. Problems. Suggested References.

6 Equations of Motion in General Relativity **266**

6.1 The Geodesic Postulate 266

Motion of a test particle. Test particle in an external gravitational field. Mass particle in gravitational field. Choice of coordinate system. Field equations. Equations of motion. Inclusion of nongravitational field. Problems.

6.2 Slow-Motion Approximation—The Einstein–Infeld–Hoffmann Equation of Motion 277

Slow-motion approximation. The double-expansion method. The approximation method. Solution of the first approximation field equations. Solution of the second approximation field equations. Remark. The equations of motion. Remarks. Problems.

6.3 Motion of Charged Particles in the Presence of Gravitation 310

The Fokker action principle. Variation of the action.

6.4 Post-Newtonian Lagrangian 316

6.5 Motion of Spinning Particles 322

Test particle with structure. The Papapetrou equations of motion. Problems.

6.6 Motion in the Schwarzschild Field—The Papapetrou–Corinaldesi Equations of Motion 328

Problems.

6.7 Motion in the Vaidya Gravitational Field 336

Geodesic motion in the Vaidya metric. Equations of motion of the spin: supplementary conditions. Derivation of the spin equations. The orbital equations. Problems.

6.8 Integrals of Motion in Particular Cases 343

6.9 Integrals of Motion in the General Case 358

Suggested References.

7 Axisymmetric Solutions of the Einstein Field Equations **365**

7.1 Stationary, Axisymmetric Metric 365

Generalization of static metric. General form of the line element.

7.2 The Papapetrou Metric 368

Lewis line element. Field equations.

7.3 The Ernst Potential 373

Field equations. The Ernst equation.

7.4 Elementary Solutions of the Ernst Equation 377

Problems.

7.5 The Kerr Metric 382
Derivation. Boyer–Lindquist coordinates.
7.6 The Tomimatsu–Sato Metric 383
7.7 The NUT–Taub Metric 385
General solutions. The Demianski–Newman metric.
7.8 Covariance Group of the Ernst Equation 388
7.9 Nonstationary Kerr Metric 389
Radiative Kerr metric. Variable-mass Kerr metric. Null tetrad quantities. Energy-momentum tensor and its asymptotic behavior.
7.10 Perturbation on the Kerr Metric Background 396
The Teukolsky master equation. Separation of the equations. Boundary conditions. Energy and polarization. Problems. Suggested References.

8 Spinor Formulation of Gravitation and Gauge Fields 407
8.1 Two-Component Spinors 407
Spinor representation of the group $SL(2, C)$. Realization of the spinor representation. Two-component spinors. Problems.
8.2 Spinors in Curved Spacetimes 415
Correspondence between spinors and tensors. Covariant derivative of a spinor. Useful formula. Problems.
8.3 The Electromagnetic Field Spinors 425
Electromagnetic potential spinor. Electromagnetic field spinor. Problems.
8.4 The Curvature Spinor 428
Spinorial Ricci identity. Symmetry of the curvature spinor. Relation to the Riemann tensor. Bianchi identities. Problems.
8.5 The Gravitational Field Spinors 434
Decomposition of the Riemann tensor. The gravitational spinor. The Ricci spinor. The Weyl spinor. The Bianchi identities. Problems.
8.6 The SU(2) Gauge Field Theory 445
Potential and field strength. Local $SU(2)$ transformation. Gauge covariant derivative. Gauge field equations. Conservation of isospin.
8.7 The Gauge Field Spinors 450
The Yang–Mills spinor. Energy-momentum spinor. $SU(2)$ spinors.
8.8 Transformation Rules for the Yang–Mills Spinors 455
General transformation properties. Transformation under rotations and boosts. Rotations around null vectors. Change of basis for spinors. Problems.

8.9 The Geometry of Gauge Fields 464

Spinor formulation. Conformal mapping of gauge fields. Problems.

8.10 The Euclidean Gauge Field Spinors 471

Algebra of the matrices s_μ. Spinor formulation of the Euclidean gauge fields. Self-dual and anti-self-dual fields. Problems. Suggested References.

9 Classification of the Gravitational and Gauge Fields 481

9.1 Classification of the Electromagnetic Field 481

Invariants of the electromagnetic field. The eigenspinor–eigenvalue equation. Classification. Problems.

9.2 Classification of the Gravitational Field 488

Properties of the Weyl tensor. Classification of the Weyl tensor. The geometry of the invariants of gravitation. The invariants in the presence of an electromagnetic field. Classification by the spinor method. Problems.

9.3 Classification of Gauge Fields: The Eigenspinor–Eigenvalue Equation 509

Invariants of the Yang–Mills field. The eigenspinor–eigenvalue equation. Problems.

9.4 The Matrix Method of Classification of SU(2) Gauge Fields 517

The electromagnetic field. SU(2) gauge fields. Problems.

9.5 Lorentz Invariant versus Gauge Invariant Methods of Classification 530

9.6 The Matrix Method of Classification—A Four-Way Scheme 532

Preliminaries. Four-way scheme of classification. Concluding remarks. Problems. Suggested References.

10 Gauge Theory of Gravitation and Other Fields 553

10.1 Differential Geometrical Analysis 553

Preliminary remarks. Differential geometry—an introduction.

10.2 Fiber Bundles and Gauge Fields 558

General relativistic interpretation of differential geometry. Fiber bundles. Abelian gauge fields. Non-Abelian gauge fields. Spinors and spacetime structure.

10.3 Fiber Bundle Foundations of the SL(2, C) Gauge Theory 562

Gauge potentials and field strengths. Free-field equations.

10.4 The SL(2, C) Theory of Gravitation 569

Coupling matter and the gauge fields. The SL(2, C) theory and the Newman–Penrose method.

10.5 Palatini-Type Variational Principle for the SL(2, C) Gauge Theory of Gravitation 572

Derivation. Remarks on quantization.

10.6 The Einstein–Maxwell Equations 579

Preliminary remarks. The electromagnetic field. Pure gravitational field equations. Combined gravitational and electromagnetic fields.

10.7 Magnetic Monopoles 590

10.8 Non-Abelian Gauge Fields in the Presence of Gravitation 593

10.9 Null Tetrad Formulation of Yang–Mills Theory 596

Yang–Mills potentials and fields. Explicit relations between potentials and fields. Yang–Mills field equations. Conserved currents. Energy-momentum tensor and the Einstein equations. Abelian solutions of the Yang–Mills theory.

10.10 Null Tetrad Formulation of the Yang–Mills Theory in Flat Spacetime 601

10.11 Monopole Solution of Yang–Mills Equations 603

Problems.

10.12 Solutions of the Coupled Einstein–Yang–Mills Field Equations 608

Problems. Suggested References.

Appendix A Extended Bodies in General Relativity 617

Index 633