



NORTH-HOLLAND

MATHEMATICS STUDIES

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Theory and Applications of Fractional Differential Equations

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