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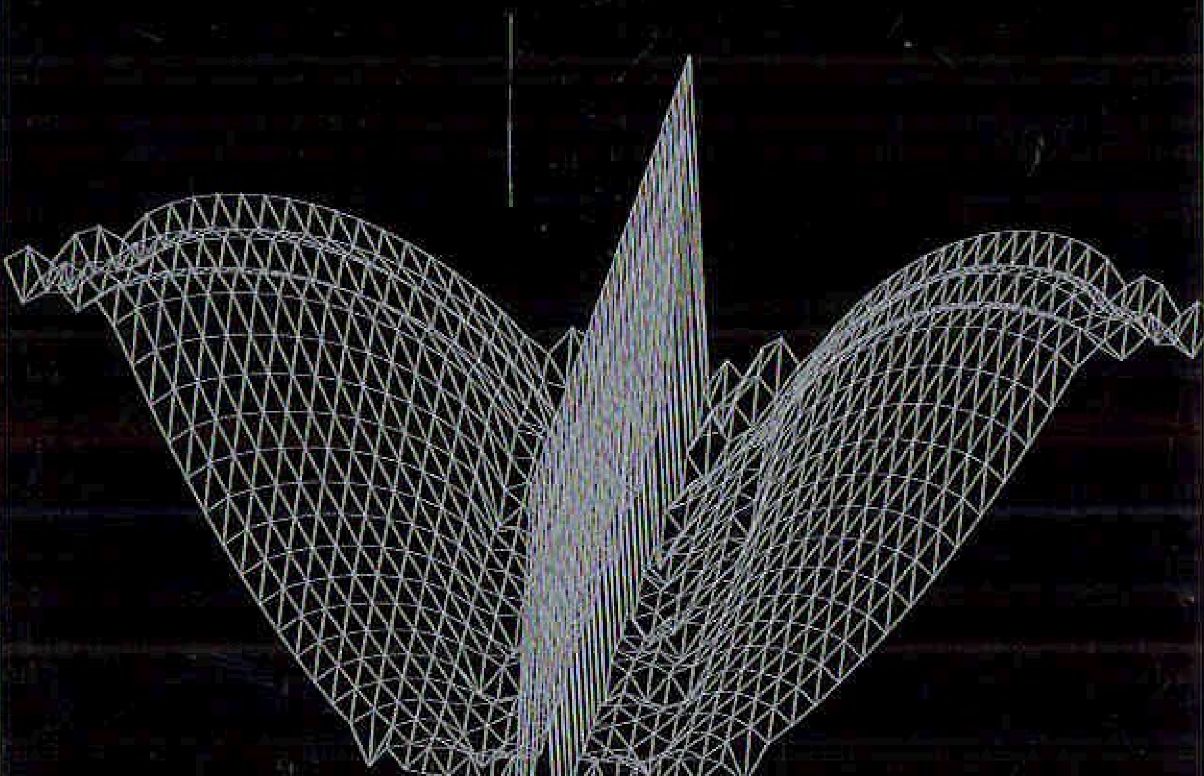
The Lanczos and Conjugate Gradient Algorithms

From Theory to Finite Precision Computations



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