

PHYSICS TEXTBOOK

 WILEY-VCH

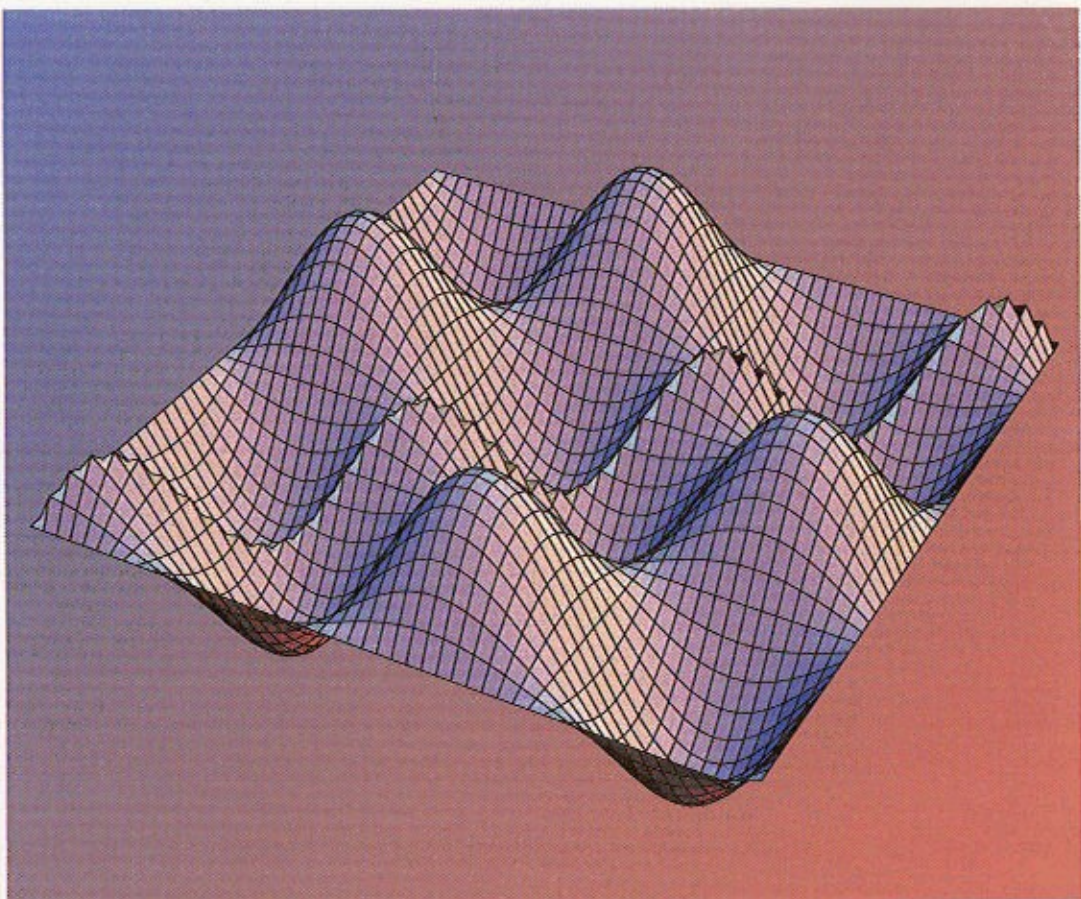
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# Graduate Mathematical Physics

With MATHEMATICA Supplements



included



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