

PATH INTEGRALS
in Quantum Mechanics,
Statistics, Polymer Physics,
and Financial Markets

4th Edition

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Contents

Preface	vii
Preface to Third Edition	ix
Preface to Second Edition	xi
Preface to First Edition	xiii
1 Fundamentals	1
1.1 Classical Mechanics	1
1.2 Relativistic Mechanics in Curved Spacetime	10
1.3 Quantum Mechanics	11
1.3.1 Bragg Reflections and Interference	12
1.3.2 Matter Waves	13
1.3.3 Schrödinger Equation	15
1.3.4 Particle Current Conservation	17
1.4 Dirac's Bra-Ket Formalism	18
1.4.1 Basis Transformations	18
1.4.2 Bracket Notation	20
1.4.3 Continuum Limit	22
1.4.4 Generalized Functions	23
1.4.5 Schrödinger Equation in Dirac Notation	25
1.4.6 Momentum States	26
1.4.7 Incompleteness and Poisson's Summation Formula	28
1.5 Observables	31
1.5.1 Uncertainty Relation	32
1.5.2 Density Matrix and Wigner Function	33
1.5.3 Generalization to Many Particles	34
1.6 Time Evolution Operator	34
1.7 Properties of Time Evolution Operator	37
1.8 Heisenberg Picture of Quantum Mechanics	39
1.9 Interaction Picture and Perturbation Expansion	42
1.10 Time Evolution Amplitude	43
1.11 Fixed-Energy Amplitude	45
1.12 Free-Particle Amplitudes	48
1.13 Quantum Mechanics of General Lagrangian Systems	51

1.14	Particle on the Surface of a Sphere	57
1.15	Spinning Top	59
1.16	Scattering	67
1.16.1	Scattering Matrix	67
1.16.2	Cross Section	68
1.16.3	Born Approximation	70
1.16.4	Partial Wave Expansion and Eikonal Approximation . . .	70
1.16.5	Scattering Amplitude from Time Evolution Amplitude . .	72
1.16.6	Lippmann-Schwinger Equation	72
1.17	Classical and Quantum Statistics	76
1.17.1	Canonical Ensemble	77
1.17.2	Grand-Canonical Ensemble	77
1.18	Density of States and Tracelog	81
Appendix 1A	Simple Time Evolution Operator	83
Appendix 1B	Convergence of Fresnel Integral	84
Appendix 1C	The Asymmetric Top	85
	Notes and References	87
2	Path Integrals — Elementary Properties and Simple Solutions	89
2.1	Path Integral Representation of Time Evolution Amplitudes . .	89
2.1.1	Sliced Time Evolution Amplitude	89
2.1.2	Zero-Hamiltonian Path Integral	91
2.1.3	Schrödinger Equation for Time Evolution Amplitude . . .	92
2.1.4	Convergence of Sliced Time Evolution Amplitude	92
2.1.5	Time Evolution Amplitude in Momentum Space	94
2.1.6	Quantum Mechanical Partition Function	96
2.1.7	Feynman's Configuration Space Path Integral	97
2.2	Exact Solution for Free Particle	101
2.2.1	Direct Solution	101
2.2.2	Fluctuations around Classical Path	102
2.2.3	Fluctuation Factor	104
2.2.4	Finite Slicing Properties of Free-Particle Amplitude	110
2.3	Exact Solution for Harmonic Oscillator	111
2.3.1	Fluctuations around Classical Path	111
2.3.2	Fluctuation Factor	113
2.3.3	The $i\eta$ -Prescription and Maslov-Morse Index	114
2.3.4	Continuum Limit	115
2.3.5	Useful Fluctuation Formulas	116
2.3.6	Oscillator Amplitude on Finite Time Lattice	118
2.4	Gelfand-Yaglom Formula	119
2.4.1	Recursive Calculation of Fluctuation Determinant	120
2.4.2	Examples	120
2.4.3	Calculation on Unsliced Time Axis	122
2.4.4	D'Alembert's Construction	123

2.4.5	Another Simple Formula	124
2.4.6	Generalization to D Dimensions	126
2.5	Harmonic Oscillator with Time-Dependent Frequency	126
2.5.1	Coordinate Space	127
2.5.2	Momentum Space	129
2.6	Free-Particle and Oscillator Wave Functions	131
2.7	General Time-Dependent Harmonic Action	133
2.8	Path Integrals and Quantum Statistics	134
2.9	Density Matrix	136
2.10	Quantum Statistics of Harmonic Oscillator	142
2.11	Time-Dependent Harmonic Potential	146
2.12	Functional Measure in Fourier Space	150
2.13	Classical Limit	153
2.14	Calculation Techniques on Sliced Time Axis. Poisson Formula	154
2.15	Field-Theoretic Definition of Harmonic Path Integral by Analytic Regularization	157
2.15.1	Zero-Temperature Evaluation of Frequency Sum	158
2.15.2	Finite-Temperature Evaluation of Frequency Sum	161
2.15.3	Tracelog of First-Order Differential Operator	163
2.15.4	Gradient Expansion of One-Dimensional Tracelog	164
2.15.5	Duality Transformation and Low-Temperature Expansion	165
2.16	Finite- N Behavior of Thermodynamic Quantities	171
2.17	Time Evolution Amplitude of Freely Falling Particle	173
2.18	Charged Particle in Magnetic Field	175
2.18.1	Action	176
2.18.2	Gauge Properties	178
2.18.3	Time-Sliced Path Integration	178
2.18.4	Classical Action	180
2.18.5	Translational Invariance	181
2.19	Charged Particle in Magnetic Field plus Harmonic Potential	182
2.20	Gauge Invariance and Alternative Path Integral Representation	184
2.21	Velocity Path Integral	185
2.22	Path Integral Representation of Scattering Matrix	186
2.22.1	General Development	186
2.22.2	Improved Formulation	189
2.22.3	Eikonal Approximation to Scattering Amplitude	190
2.23	Heisenberg Operator Approach to Time Evolution Amplitude	190
2.23.1	Free Particle	191
2.23.2	Harmonic Oscillator	193
2.23.3	Charged Particle in Magnetic Field	193
Appendix 2A	Baker-Campbell-Hausdorff Formula and Magnus Expansion	197
Appendix 2B	Direct Calculation of Time-Sliced Oscillator Amplitude	200
Appendix 2C	Derivation of Mehler Formula	201

Notes and References	202
3 External Sources, Correlations, and Perturbation Theory	205
3.1 External Sources	205
3.2 Green Function of Harmonic Oscillator	209
3.2.1 Wronski Construction	209
3.2.2 Spectral Representation	213
3.3 Green Functions of First-Order Differential Equation	215
3.3.1 Time-Independent Frequency	215
3.3.2 Time-Dependent Frequency	222
3.4 Summing Spectral Representation of Green Function	225
3.5 Wronski Construction for Periodic and Antiperiodic Green Functions	227
3.6 Time Evolution Amplitude in Presence of Source Term	228
3.7 Time Evolution Amplitude at Fixed Path Average	232
3.8 External Source in Quantum-Statistical Path Integral	233
3.8.1 Continuation of Real-Time Result	234
3.8.2 Calculation at Imaginary Time	238
3.9 Lattice Green Function	245
3.10 Correlation Functions, Generating Functional, and Wick Expansion	245
3.10.1 Real-Time Correlation Functions	248
3.11 Correlation Functions of Charged Particle in Magnetic Field	250
3.12 Correlation Functions in Canonical Path Integral	251
3.12.1 Harmonic Correlation Functions	252
3.12.2 Relations between Various Amplitudes	254
3.12.3 Harmonic Generating Functionals	255
3.13 Particle in Heat Bath	258
3.14 Heat Bath of Photons	262
3.15 Harmonic Oscillator in Ohmic Heat Bath	264
3.16 Harmonic Oscillator in Photon Heat Bath	267
3.17 Perturbation Expansion of Anharmonic Systems	268
3.18 Rayleigh-Schrödinger and Brillouin-Wigner Perturbation Expansion	272
3.19 Level-Shifts and Perturbed Wave Functions from Schrödinger Equation	276
3.20 Calculation of Perturbation Series via Feynman Diagrams	278
3.21 Perturbative Definition of Interacting Path Integrals	283
3.22 Generating Functional of Connected Correlation Functions	284
3.22.1 Connectedness Structure of Correlation Functions	285
3.22.2 Correlation Functions versus Connected Correlation Functions	288
3.22.3 Functional Generation of Vacuum Diagrams	290
3.22.4 Correlation Functions from Vacuum Diagrams	294
3.22.5 Generating Functional for Vertex Functions. Effective Action	296
3.22.6 Ginzburg-Landau Approximation to Generating Functional	301

3.22.7	Composite Fields	302
3.23	Path Integral Calculation of Effective Action by Loop Expansion	303
3.23.1	General Formalism	303
3.23.2	Mean-Field Approximation	304
3.23.3	Corrections from Quadratic Fluctuations	308
3.23.4	Effective Action to Second Order in $\hbar$	311
3.23.5	Finite-Temperature Two-Loop Effective Action	315
3.23.6	Background Field Method for Effective Action	317
3.24	Nambu-Goldstone Theorem	320
3.25	Effective Classical Potential	322
3.25.1	Effective Classical Boltzmann Factor	323
3.25.2	Effective Classical Hamiltonian	326
3.25.3	High- and Low-Temperature Behavior	327
3.25.4	Alternative Candidate for Effective Classical Potential	328
3.25.5	Harmonic Correlation Function without Zero Mode	329
3.25.6	Perturbation Expansion	330
3.25.7	First-Order Perturbative Result	332
3.26	Perturbative Approach to Scattering Amplitude	334
3.26.1	Generating Functional	334
3.26.2	Application to Scattering Amplitude	335
3.26.3	First Correction to Eikonal Approximation	335
3.26.4	Rayleigh-Schrödinger Expansion of Scattering Amplitude	336
3.27	Functional Determinants from Green Functions	338
Appendix 3A	Matrix Elements for General Potential	344
Appendix 3B	Energy Shifts for $gx^4/4$ -Interaction	345
Appendix 3C	Recursion Relations for Perturbation Coefficients	347
3C.1	One-Dimensional Interaction x^4	347
3C.2	One-Dimensional Interactions x^4 and x^3	350
3C.3	Ground-State Energy with External Current	352
3C.4	Recursion Relation for Effective Potential	353
3C.5	Interaction r^4 in D -Dimensional Radial Oscillator	356
3C.6	Interaction r^{2q} in D Dimensions	357
3C.7	Polynomial Interaction in D Dimensions	358
Appendix 3D	Feynman Integrals for $T \neq 0$	358
	Notes and References	361
4	Semiclassical Time Evolution Amplitude	363
4.1	Wentzel-Kramers-Brillouin (WKB) Approximation	363
4.2	Saddle Point Approximation	368
4.2.1	Ordinary Integrals	368
4.2.2	Path Integrals	371
4.3	Van Vleck-Pauli-Morette Determinant	377
4.4	Fundamental Composition Law for Semiclassical Time Evolution Amplitude	381

4.5	Semiclassical Fixed-Energy Amplitude	383
4.6	Semiclassical Amplitude in Momentum Space	385
4.7	Semiclassical Quantum-Mechanical Partition Function	387
4.8	Multi-Dimensional Systems	393
4.9	Quantum Corrections to Classical Density of States	397
4.9.1	One-Dimensional Case	398
4.9.2	Arbitrary Dimensions	399
4.9.3	Bilocal Density of States	401
4.9.4	Gradient Expansion of Tracelog of Hamiltonian Operator	403
4.9.5	Local Density of States on Circle	407
4.9.6	Quantum Corrections to Bohr-Sommerfeld Approximation	408
4.10	Thomas-Fermi Model of Neutral Atoms	410
4.10.1	Semiclassical Limit	410
4.10.2	Self-Consistent Field Equation	412
4.10.3	Energy Functional of Thomas-Fermi Atom	414
4.10.4	Calculation of Energies	415
4.10.5	Virial Theorem	418
4.10.6	Exchange Energy	419
4.10.7	Quantum Correction Near Origin	420
4.10.8	Systematic Quantum Corrections to Thomas-Fermi Energies	423
4.11	Classical Action of Coulomb System	427
4.12	Semiclassical Scattering	435
4.12.1	General Formulation	435
4.12.2	Semiclassical Cross Section of Mott Scattering	439
Appendix 4A	Semiclassical Quantization for Pure Power Potentials	440
Appendix 4B	Derivation of Semiclassical Time Evolution Amplitude	442
	Notes and References	446
5	Variational Perturbation Theory	448
5.1	Variational Approach to Effective Classical Partition Function	448
5.2	Local Harmonic Trial Partition Function	449
5.3	Optimal Upper Bound	454
5.4	Accuracy of Variational Approximation	455
5.5	Weakly Bound Ground State Energy in Finite-Range Potential Well	457
5.6	Possible Direct Generalizations	459
5.7	Effective Classical Potential for Anharmonic Oscillator	460
5.8	Particle Densities	466
5.9	Extension to D Dimensions	469
5.10	Application to Coulomb and Yukawa Potentials	471
5.11	Hydrogen Atom in Strong Magnetic Field	474
5.11.1	Weak-Field Behavior	477
5.11.2	Effective Classical Hamiltonian	478
5.12	Effective Potential and Magnetization Curves	481
5.13	Variational Approach to Excitation Energies	483

5.14	Systematic Improvement of Feynman-Kleinert Approximation . . .	487
5.15	Applications of Variational Perturbation Expansion	490
5.15.1	Anharmonic Oscillator at $T = 0$	490
5.15.2	Anharmonic Oscillator for $T > 0$	492
5.16	Convergence of Variational Perturbation Expansion	496
5.17	Variational Perturbation Theory for Strong-Coupling Expansion	503
5.18	General Strong-Coupling Expansions	506
5.19	Variational Interpolation between Weak and Strong-Coupling Ex- pansions	509
5.20	Systematic Improvement of Excited Energies	510
5.21	Variational Treatment of Double-Well Potential	511
5.22	Higher-Order Effective Classical Potential for Nonpolynomial In- teractions	512
5.22.1	Evaluation of Path Integrals	514
5.22.2	Higher-Order Smearing Formula in D Dimensions	515
5.22.3	Isotropic Second-Order Approximation to Coulomb Problem	517
5.22.4	Anisotropic Second-Order Approximation to Coulomb Prob- lem	518
5.22.5	Zero-Temperature Limit	519
5.23	Polarons	523
5.23.1	Partition Function	525
5.23.2	Harmonic Trial System	527
5.23.3	Effective Mass	533
5.23.4	Second-Order Correction	533
5.23.5	Polaron in Magnetic Field, Bipolarons, etc.	534
5.23.6	Variational Interpolation for Polaron Energy and Mass . .	535
5.24	Density Matrices	538
5.24.1	Harmonic Oscillator	539
5.24.2	Variational Perturbation Theory for Density Matrices . . .	540
5.24.3	Smearing Formula for Density Matrices	542
5.24.4	First-Order Variational Approximation	544
5.24.5	Smearing Formula in Higher Spatial Dimensions	549
5.24.6	Applications	550
Appendix 5A	Feynman Integrals for $T \neq 0$ without Zero Frequency .	560
Appendix 5B	Proof of Scaling Relation for the Extrema of W_N	562
Appendix 5C	Second-Order Shift of Polaron Energy	564
	Notes and References	565
6	Path Integrals with Topological Constraints	571
6.1	Point Particle on Circle	571
6.2	Infinite Wall	575
6.3	Point Particle in Box	579
6.4	Strong-Coupling Theory for Particle in Box	582
6.4.1	Partition Function	583

6.4.2	Perturbation Expansion	583
6.4.3	Variational Strong-Coupling Approximations	585
6.4.4	Special Properties of Expansion	587
6.4.5	Exponentially Fast Convergence	588
	Notes and References	589
7	Many Particle Orbits — Statistics and Second Quantization	591
7.1	Ensembles of Bose and Fermi Particle Orbits	592
7.2	Bose-Einstein Condensation	599
7.2.1	Free Bose Gas	599
7.2.2	Effect of Interactions	609
7.2.3	Bose-Einstein Condensation in Harmonic Trap	614
7.2.4	Entropy and Specific Heat	622
7.2.5	Interactions in Harmonic Trap	624
7.3	Gas of Free Fermions	628
7.4	Statistics Interaction	634
7.5	Fractional Statistics	638
7.6	Second-Quantized Bose Fields	640
7.7	Fluctuating Bose Fields	643
7.8	Coherent States	649
7.9	Second-Quantized Fermi Fields	652
7.10	Fluctuating Fermi Fields	653
7.10.1	Grassmann Variables	653
7.10.2	Fermionic Functional Determinant	656
7.10.3	Coherent States for Fermions	659
7.11	Hilbert Space of Quantized Grassmann Variable	661
7.11.1	Single Real Grassmann Variable	661
7.11.2	Quantizing Harmonic Oscillator with Grassmann Variables	665
7.11.3	Spin System with Grassmann Variables	666
7.12	External Sources in a^*, a -Path Integral	670
7.13	Generalization to Pair Terms	672
7.14	Spatial Degrees of Freedom	674
7.14.1	Grand-Canonical Ensemble of Particle Orbits from Free Fluctuating Field	675
7.14.2	First versus Second Quantization	677
7.14.3	Interacting Fields	677
7.14.4	Effective Classical Field Theory	678
7.15	Bosonization	680
7.15.1	Collective Field	681
7.15.2	Bosonized versus Original Theory	683
	Notes and References	685

8	Path Integrals in Polar and Spherical Coordinates	690
8.1	Angular Decomposition in Two Dimensions	690
8.2	Trouble with Feynman's Path Integral Formula in Radial Coordinates	693
8.3	Cautionary Remarks	697
8.4	Time Slicing Corrections	700
8.5	Angular Decomposition in Three and More Dimensions	704
8.5.1	Three Dimensions	705
8.5.2	D Dimensions	707
8.6	Radial Path Integral for Harmonic Oscillator and Free Particle	713
8.7	Particle near the Surface of a Sphere in D Dimensions	714
8.8	Angular Barriers near the Surface of a Sphere	717
8.8.1	Angular Barriers in Three Dimensions	717
8.8.2	Angular Barriers in Four Dimensions	722
8.9	Motion on a Sphere in D Dimensions	727
8.10	Path Integrals on Group Spaces	731
8.11	Path Integral of Spinning Top	734
8.12	Path Integral of Spinning Particle	735
8.13	Berry Phase	739
8.14	Spin Precession	740
	Notes and References	741
9	Wave Functions	744
9.1	Free Particle in D Dimensions	744
9.2	Harmonic Oscillator in D Dimensions	747
9.3	Free Particle from $\omega \rightarrow 0$ -Limit of Oscillator	753
9.4	Charged Particle in Uniform Magnetic Field	755
9.5	Dirac δ -Function Potential	762
	Notes and References	764
10	Spaces with Curvature and Torsion	765
10.1	Einstein's Equivalence Principle	766
10.2	Classical Motion of Mass Point in General Metric-Affine Space	767
10.2.1	Equations of Motion	767
10.2.2	Nonholonomic Mapping to Spaces with Torsion	770
10.2.3	New Equivalence Principle	776
10.2.4	Classical Action Principle for Spaces with Curvature and Torsion	776
10.3	Path Integral in Metric-Affine Space	781
10.3.1	Nonholonomic Transformation of Action	781
10.3.2	Measure of Path Integration	786
10.4	Completing Solution of Path Integral on Surface of Sphere	792
10.5	External Potentials and Vector Potentials	794
10.6	Perturbative Calculation of Path Integrals in Curved Space	796

10.6.1	Free and Interacting Parts of Action	796
10.6.2	Zero Temperature	799
10.7	Model Study of Coordinate Invariance	801
10.7.1	Diagrammatic Expansion	803
10.7.2	Diagrammatic Expansion in d Time Dimensions	805
10.8	Calculating Loop Diagrams	806
10.8.1	Reformulation in Configuration Space	813
10.8.2	Integrals over Products of Two Distributions	814
10.8.3	Integrals over Products of Four Distributions	815
10.9	Distributions as Limits of Bessel Function	817
10.9.1	Correlation Function and Derivatives	817
10.9.2	Integrals over Products of Two Distributions	819
10.9.3	Integrals over Products of Four Distributions	820
10.10	Simple Rules for Calculating Singular Integrals	822
10.11	Perturbative Calculation on Finite Time Intervals	827
10.11.1	Diagrammatic Elements	828
10.11.2	Cumulant Expansion of D -Dimensional Free-Particle Amplitude in Curvilinear Coordinates	829
10.11.3	Propagator in $1 - \varepsilon$ Time Dimensions	831
10.11.4	Coordinate Independence for Dirichlet Boundary Conditions	832
10.11.5	Time Evolution Amplitude in Curved Space	838
10.11.6	Covariant Results for Arbitrary Coordinates	844
10.12	Effective Classical Potential in Curved Space	849
10.12.1	Covariant Fluctuation Expansion	850
10.12.2	Arbitrariness of q_0^μ	853
10.12.3	Zero-Mode Properties	854
10.12.4	Covariant Perturbation Expansion	857
10.12.5	Covariant Result from Noncovariant Expansion	858
10.12.6	Particle on Unit Sphere	861
10.13	Covariant Effective Action for Quantum Particle with Coordinate-Dependent Mass	863
10.13.1	Formulating the Problem	864
10.13.2	Gradient Expansion	867
Appendix 10A	Nonholonomic Gauge Transformations in Electromagnetism	867
10A.1	Gradient Representation of Magnetic Field of Current Loops	868
10A.2	Generating Magnetic Fields by Multivalued Gauge Transformations	872
10A.3	Magnetic Monopoles	873
10A.4	Minimal Magnetic Coupling of Particles from Multivalued Gauge Transformations	875
10A.5	Gauge Field Representation of Current Loops and Monopoles	876
Appendix 10B	Comparison of Multivalued Basis Tetrads with Vierbein Fields	878

Appendix 10C	Cancellation of Powers of $\delta(0)$	880
	Notes and References	882
11	Schrödinger Equation in General Metric-Affine Spaces	886
11.1	Integral Equation for Time Evolution Amplitude	886
11.1.1	From Recursion Relation to Schrödinger Equation	887
11.1.2	Alternative Evaluation	890
11.2	Equivalent Path Integral Representations	893
11.3	Potentials and Vector Potentials	897
11.4	Unitarity Problem	898
11.5	Alternative Attempts	901
11.6	DeWitt-Seeley Expansion of Time Evolution Amplitude	902
Appendix 11A	Cancellations in Effective Potential	905
Appendix 11B	DeWitt's Amplitude	908
	Notes and References	908
12	New Path Integral Formula for Singular Potentials	910
12.1	Path Collapse in Feynman's formula for the Coulomb System	910
12.2	Stable Path Integral with Singular Potentials	913
12.3	Time-Dependent Regularization	918
12.4	Relation to Schrödinger Theory. Wave Functions	920
	Notes and References	922
13	Path Integral of Coulomb System	923
13.1	Pseudotime Evolution Amplitude	923
13.2	Solution for the Two-Dimensional Coulomb System	925
13.3	Absence of Time Slicing Corrections for $D = 2$	930
13.4	Solution for the Three-Dimensional Coulomb System	935
13.5	Absence of Time Slicing Corrections for $D = 3$	941
13.6	Geometric Argument for Absence of Time Slicing Corrections	943
13.7	Comparison with Schrödinger Theory	944
13.8	Angular Decomposition of Amplitude, and Radial Wave Functions	949
13.9	Remarks on Geometry of Four-Dimensional u^μ -Space	953
13.10	Solution in Momentum Space	955
13.10.1	Gauge-Invariant Canonical Path Integral	956
13.10.2	Another Form of Action	959
13.10.3	Absence of Extra R -Term	960
Appendix 13A	Dynamical Group of Coulomb States	960
	Notes and References	964
14	Solution of Further Path Integrals by Duru-Kleinert Method	966
14.1	One-Dimensional Systems	966
14.2	Derivation of the Effective Potential	970
14.3	Comparison with Schrödinger Quantum Mechanics	974

14.4	Applications	975
14.4.1	Radial Harmonic Oscillator and Morse System	975
14.4.2	Radial Coulomb System and Morse System	977
14.4.3	Equivalence of Radial Coulomb System and Radial Oscillator	978
14.4.4	Angular Barrier near Sphere, and Rosen-Morse Potential	986
14.4.5	Angular Barrier near Four-Dimensional Sphere, and General Rosen-Morse Potential	988
14.4.6	Hulthén Potential and General Rosen-Morse Potential	991
14.4.7	Extended Hulthén Potential and General Rosen-Morse Potential	994
14.5	D -Dimensional Systems	994
14.6	Path Integral of the Dionium Atom	996
14.6.1	Formal Solution	997
14.6.2	Absence of Time Slicing Corrections	1001
14.7	Time-Dependent Duru-Kleinert Transformation	1004
Appendix 14A	Affine Connection of Dionium Atom	1007
Appendix 14B	Algebraic Aspects of Dionium States	1008
	Notes and References	1008
15	Path Integrals in Polymer Physics	1010
15.1	Polymers and Ideal Random Chains	1010
15.2	Moments of End-to-End Distribution	1012
15.3	Exact End-to-End Distribution in Three Dimensions	1015
15.4	Short-Distance Expansion for Long Polymer	1017
15.5	Saddle Point Approximation to Three-Dimensional End-to-End Distribution	1019
15.6	Path Integral for Continuous Gaussian Distribution	1020
15.7	Stiff Polymers	1023
15.7.1	Sliced Path Integral	1025
15.7.2	Relation to Classical Heisenberg Model	1026
15.7.3	End-to-End Distribution	1028
15.7.4	Moments of End-to-End Distribution	1028
15.8	Continuum Formulation	1029
15.8.1	Path Integral	1029
15.8.2	Correlation Functions and Moments	1030
15.9	Schrödinger Equation and Recursive Solution for Moments	1034
15.9.1	Setting up the Schrödinger Equation	1034
15.9.2	Recursive Solution of Schrödinger Equation	1035
15.9.3	From Moments to End-to-End Distribution for $D = 3$	1038
15.9.4	Large-Stiffness Approximation to End-to-End Distribution	1040
15.9.5	Higher Loop Corrections	1046
15.10	Excluded-Volume Effects	1053
15.11	Flory's Argument	1061

15.12	Polymer Field Theory	1062
15.13	Fermi Fields for Self-Avoiding Lines	1069
Appendix 15A	Basic Integrals	1070
Appendix 15B	Loop Integrals	1070
Appendix 15C	Integrals Involving Modified Green Function	1072
	Notes and References	1073
16	Polymers and Particle Orbits in Multiply Connected Spaces	1076
16.1	Simple Model for Entangled Polymers	1076
16.2	Entangled Fluctuating Particle Orbit: Aharonov-Bohm Effect	1080
16.3	Aharonov-Bohm Effect and Fractional Statistics	1089
16.4	Self-Entanglement of Polymer	1094
16.5	The Gauss Invariant of Two Curves	1108
16.6	Bound States of Polymers and Ribbons	1111
16.7	Chern-Simons Theory of Entanglements	1117
16.8	Entangled Pair of Polymers	1121
16.8.1	Polymer Field Theory for Probabilities	1123
16.8.2	Calculation of Partition Function	1124
16.8.3	Calculation of Numerator in Second Moment	1126
16.8.4	First Diagram in Fig. 16.23	1127
16.8.5	Second and Third Diagrams in Fig. 16.23	1128
16.8.6	Fourth Diagram in Fig. 16.23	1129
16.8.7	Second Topological Moment	1130
16.9	Chern-Simons Theory of Statistical Interaction	1131
16.10	Second-Quantized Anyon Fields	1134
16.11	Fractional Quantum Hall Effect	1137
16.12	Anyonic Superconductivity	1140
16.13	Non-Abelian Chern-Simons Theory	1142
Appendix 16A	Calculation of Feynman Diagrams in Polymer Entangle- ment	1145
Appendix 16B	Kauffman and BLM/Ho polynomials	1146
Appendix 16C	Skein Relation between Wilson Loop Integrals	1147
Appendix 16D	London Equations	1150
Appendix 16E	Hall Effect in Electron Gas	1151
	Notes and References	1151
17	Tunneling	1157
17.1	Double-Well Potential	1157
17.2	Classical Solutions — Kinks and Antikinks	1160
17.3	Quadratic Fluctuations	1164
17.3.1	Zero-Eigenvalue Mode	1170
17.3.2	Continuum Part of Fluctuation Factor	1174
17.4	General Formula for Eigenvalue Ratios	1176
17.5	Fluctuation Determinant from Classical Solution	1178

17.6	Wave Functions of Double-Well	1182
17.7	Gas of Kinks and Antikinks and Level Splitting Formula	1183
17.8	Fluctuation Correction to Level Splitting	1187
17.9	Tunneling and Decay	1192
17.10	Large-Order Behavior of Perturbation Expansions	1200
17.10.1	Growth Properties of Expansion Coefficients	1201
17.10.2	Semiclassical Large-Order Behavior	1204
17.10.3	Fluctuation Correction to the Imaginary Part and Large-Order Behavior	1209
17.10.4	Variational Approach to Tunneling. Perturbation Coefficients to All Orders	1212
17.10.5	Convergence of Variational Perturbation Expansion	1220
17.11	Decay of Supercurrent in Thin Closed Wire	1228
17.12	Decay of Metastable Thermodynamic Phases	1240
17.13	Decay of Metastable Vacuum State in Quantum Field Theory	1247
17.14	Crossover from Quantum Tunneling to Thermally Driven Decay	1248
Appendix 17A	Feynman Integrals for Fluctuation Correction	1250
	Notes and References	1252

18 Nonequilibrium Quantum Statistics 1255

18.1	Linear Response and Time-Dependent Green Functions for $T \neq 0$	1255
18.2	Spectral Representations of Green Functions for $T \neq 0$	1258
18.3	Other Important Green Functions	1261
18.4	Hermitian Adjoint Operators	1264
18.5	Harmonic Oscillator Green Functions for $T \neq 0$	1265
18.5.1	Creation Annihilation Operators	1265
18.5.2	Real Field Operators	1268
18.6	Nonequilibrium Green Functions	1270
18.7	Perturbation Theory for Nonequilibrium Green Functions	1279
18.8	Path Integral Coupled to Thermal Reservoir	1282
18.9	Fokker-Planck Equation	1288
18.9.1	Canonical Path Integral for Probability Distribution	1289
18.9.2	Solving the Operator Ordering Problem	1290
18.9.3	Strong Damping	1296
18.10	Langevin Equations	1299
18.11	Stochastic Quantization	1303
18.12	Stochastic Calculus	1306
18.13	Supersymmetry	1312
18.14	Stochastic Quantum Liouville Equation	1315
18.15	Master Equation for Time Evolution	1317
18.16	Relation to Quantum Langevin Equation	1320
18.17	Electromagnetic Dissipation and Decoherence	1320
18.17.1	Forward-Backward Path Integral	1321
18.17.2	Master Equation for Time Evolution in Photon Bath	1325

18.17.3	Line Width	1326
18.17.4	Lamb shift	1328
18.17.5	Langevin Equations	1331
18.18	Fokker-Planck Equation in Spaces with Curvature and Torsion .	1333
18.19	Stochastic Interpretation of Quantum-Mechanical Amplitudes .	1334
18.20	Stochastic Equation for Schrödinger Wave Function	1336
18.21	Real Stochastic and Deterministic Equation for Schrödinger Wave Function	1338
18.21.1	Stochastic Differential Equation	1339
18.21.2	Equation for Noise Average	1339
18.21.3	Harmonic Oscillator	1340
18.21.4	General Potential	1341
18.21.5	Deterministic Equation	1342
18.22	Heisenberg Picture for Probability Evolution	1342
Appendix 18A	Inequalities for Diagonal Green Functions	1346
Appendix 18B	General Generating Functional	1349
Appendix 18C	Wick Decomposition of Operator Products	1354
	Notes and References	1355
19	Relativistic Particle Orbits	1359
19.1	Special Features of Relativistic Path Integrals	1361
19.2	Proper Action for Fluctuating Relativistic Particle Orbits	1364
19.2.1	Gauge-Invariant Formulation	1364
19.2.2	Simplest Gauge Fixing	1366
19.2.3	Partition Function of Ensemble of Closed Particle Loops .	1368
19.2.4	Fixed-Energy Amplitude	1369
19.3	Tunneling in Relativistic Physics	1370
19.3.1	Decay Rate of Vacuum in Electric Field	1370
19.3.2	Birth of Universe	1379
19.3.3	Friedmann Model	1384
19.3.4	Tunneling of Expanding Universe	1388
19.4	Relativistic Coulomb System	1388
19.5	Relativistic Particle in Electromagnetic Field	1392
19.5.1	Action and Partition Function	1392
19.5.2	Perturbation Expansion	1393
19.5.3	Lowest-Order Vacuum Polarization	1396
19.6	Path Integral for Spin-1/2 Particle	1400
19.6.1	Dirac Theory	1400
19.6.2	Path Integral	1404
19.6.3	Amplitude with Electromagnetic Interaction	1406
19.6.4	Effective Action in Electromagnetic Field	1409
19.6.5	Perturbation Expansion	1410
19.6.6	Vacuum Polarization	1411
19.7	Supersymmetry	1413

19.7.1	Global Invariance	1413
19.7.2	Local Invariance	1414
	Notes and References	1416
20	Path Integrals and Financial Markets	1418
20.1	Fluctuation Properties of Financial Assets	1418
20.1.1	Harmonic Approximation to Fluctuations	1420
20.1.2	Lévy Distributions	1422
20.1.3	Truncated Lévy Distributions	1423
20.1.4	Student or Tsallis Distribution	1428
20.1.5	Asymmetric Truncated Lévy Distributions	1430
20.1.6	Boltzmann Distribution	1432
20.1.7	Meixner Distributions	1434
20.1.8	Generalized Hyperbolic Distributions	1436
20.1.9	Lévy-Khintchine Formula	1438
20.1.10	Debye-Waller Factor for Non-Gaussian Fluctuations	1439
20.1.11	Path Integral for Non-Gaussian Distribution	1439
20.1.12	Semigroup Property of Asset Distributions	1441
20.1.13	Time Evolution of Distribution	1442
20.1.14	Fokker-Planck-Type Equation	1448
20.2	Martingales	1451
20.2.1	Gaussian Martingales	1451
20.2.2	Non-Gaussian Martingales	1452
20.3	Origin of Heavy Tails	1454
20.3.1	Pair of Stochastic Differential Equations	1454
20.3.2	Fokker-Planck Equation	1455
20.3.3	Solution of Fokker-Planck Equation	1458
20.3.4	Pure x -Distribution	1459
20.3.5	Long-Time Behavior	1462
20.3.6	Tail Behavior for all Times	1465
20.3.7	Path Integral Calculation	1467
20.3.8	Natural Martingales	1468
20.4	Spectral Decomposition of Power Behaviors	1469
20.5	Option Pricing	1470
20.5.1	Black-Scholes Option Pricing Model	1471
20.5.2	Evolution Equations of Portfolios with Options	1473
20.5.3	Option Pricing for Gaussian Fluctuations	1476
20.5.4	Option Pricing for Boltzmann Distribution	1478
20.5.5	Option Pricing for General Non-Gaussian Fluctuations	1480
20.5.6	Option Pricing for Fluctuating Variance	1483
20.5.7	Perturbation Expansion and Smile	1485
Appendix 20A	Large- x Behavior of Truncated Lévy Distribution	1488
Appendix 20B	Gaussian Weight	1490
Appendix 20C	Comparison with Dow-Jones Data	1491

Notes and References	1492
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Index	1499
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