

# Linear Systems

Optimal and Robust Control

Alok Sinha



CRC Press  
Taylor & Francis Group

# Contents

|                  |                                                                                    |    |
|------------------|------------------------------------------------------------------------------------|----|
| <b>Chapter 1</b> | Introduction .....                                                                 | 1  |
| 1.1              | Overview .....                                                                     | 1  |
| 1.2              | Contents of the Book .....                                                         | 2  |
| <b>Chapter 2</b> | State Space Description of a Linear System .....                                   | 3  |
| 2.1              | Transfer Function of a Single Input/Single Output (SISO) System .....              | 3  |
| 2.2              | State Space Realizations of a SISO System .....                                    | 5  |
|                  | Method I .....                                                                     | 5  |
|                  | Method II .....                                                                    | 7  |
|                  | Properties of State Space Models .....                                             | 10 |
|                  | 1. Duality .....                                                                   | 10 |
|                  | 2. Nonuniqueness of State Space Realization .....                                  | 10 |
| 2.3              | SISO Transfer Function from a State Space Realization .....                        | 11 |
| 2.4              | Solution of State Space Equations .....                                            | 12 |
|                  | 2.4.1 Homogeneous Equation .....                                                   | 12 |
|                  | 2.4.2 Inhomogeneous Equation .....                                                 | 13 |
| 2.5              | Observability and Controllability of a SISO System .....                           | 14 |
|                  | 2.5.1 Observability .....                                                          | 14 |
|                  | Observability of State Space Realization Using Method I .....                      | 16 |
|                  | Observability of State Space Realization Using Method II .....                     | 16 |
|                  | 2.5.2 Controllability .....                                                        | 19 |
|                  | Controllability of the State Space Realization Obtained<br>Using Method I .....    | 24 |
|                  | Controllability of the State Space Realization Obtained<br>Using Method II .....   | 25 |
| 2.6              | Some Important Similarity Transformations .....                                    | 29 |
|                  | 2.6.1 Diagonal Form .....                                                          | 29 |
|                  | 2.6.2 Controllability Canonical Form .....                                         | 30 |
| 2.7              | Simultaneous Controllability and Observability .....                               | 31 |
|                  | 2.7.1 Observability of State Space Realization Using Method I .....                | 32 |
| 2.8              | Multiinput/Multioutput (MIMO) Systems .....                                        | 36 |
| 2.9              | State Space Realizations of a Transfer Function Matrix .....                       | 40 |
|                  | Method I .....                                                                     | 40 |
|                  | Method II .....                                                                    | 42 |
| 2.10             | Controllability and Observability of a MIMO System .....                           | 44 |
|                  | 2.10.1 Controllability and Observability of Methods I and II<br>Realizations ..... | 45 |
|                  | Method I Realization .....                                                         | 45 |

|                                                                                                                                   |    |
|-----------------------------------------------------------------------------------------------------------------------------------|----|
| Method II Realization .....                                                                                                       | 46 |
| 2.11 Matrix-Fraction Description (MFD) .....                                                                                      | 46 |
| 2.11.1 Degree of a Square Polynomial Matrix and Greatest<br>Common Right Divisor ( <i>gcd</i> ) .....                             | 47 |
| 2.11.2 Elementary Row and Column Operations .....                                                                                 | 50 |
| 2.11.3 Determination of a <i>gcd</i> .....                                                                                        | 51 |
| 2.12 MFD of a Transfer Function Matrix for the Minimal Order of a State<br>Space Realization .....                                | 57 |
| 2.13 Controller Form Realization from a Right MFD .....                                                                           | 59 |
| 2.13.1 State Space Realization .....                                                                                              | 59 |
| 2.13.2 Similarity Transformation to Convert Any State Space<br>Realization $\{A, B, C\}$ to the Controller Form Realization ..... | 64 |
| 2.14 Poles and Zeros of a MIMO Transfer Function Matrix .....                                                                     | 66 |
| 2.14.1 Smith Form .....                                                                                                           | 66 |
| 2.14.2 Smith–McMillan Form .....                                                                                                  | 67 |
| 2.14.3 Poles and Zeros via Smith–McMillan Form .....                                                                              | 69 |
| 2.14.4 Poles and Zeros via an Irreducible MFD .....                                                                               | 71 |
| 2.15 Stability Analysis .....                                                                                                     | 71 |
| An Important Property of the Lyapunov Equation .....                                                                              | 72 |
| Exercise Problems .....                                                                                                           | 73 |

### **Chapter 3 State Feedback Control and Optimization .....** 79

|                                                                                                                                      |     |
|--------------------------------------------------------------------------------------------------------------------------------------|-----|
| 3.1 State Variable Feedback for a Single Input System .....                                                                          | 79  |
| 3.1.1 Effects of State Feedback on Poles of the Closed-Loop<br>Transfer Function: Computation of State Feedback Gain<br>Vector ..... | 80  |
| 3.1.2 Effects of State Feedback on Zeros of the Closed-Loop<br>Transfer Function .....                                               | 85  |
| 3.1.3 State Feedback Control for a Nonzero and Constant Output .....                                                                 | 89  |
| 3.1.4 State Feedback Control under Constant Input Disturbances:<br>Integral Action .....                                             | 91  |
| 3.2 Computation of State Feedback Gain Matrix for a Multiinput System .....                                                          | 95  |
| 3.3 State Feedback Gain Matrix for a Multiinput System for Desired<br>Eigenvalues and Eigenvectors .....                             | 99  |
| 3.4 Fundamentals of Optimal Control Theory .....                                                                                     | 112 |
| 3.4.1 Necessary Conditions for Optimality .....                                                                                      | 113 |
| Minimization of $J$ .....                                                                                                            | 114 |
| Maximization of $J$ .....                                                                                                            | 115 |
| 3.4.2 Properties of Hamiltonian for an Autonomous System .....                                                                       | 115 |
| Special Case .....                                                                                                                   | 116 |
| 3.5 Linear Quadratic Regulator (LQR) Problem .....                                                                                   | 123 |
| 3.5.1 Solution (Open-Loop Optimal Control) .....                                                                                     | 124 |
| 3.5.2 Solution (Closed-Loop Optimal Control) .....                                                                                   | 126 |
| 3.5.3 Cross Term in the Objective Function .....                                                                                     | 127 |
| 3.5.4 Important Case: Infinite Final Time .....                                                                                      | 132 |

|                  |                                                                                                                                        |            |
|------------------|----------------------------------------------------------------------------------------------------------------------------------------|------------|
| 3.6              | Solution of LQR Problem Via Root Locus Plot: SISO Case .....                                                                           | 137        |
|                  | The Symmetric Root Locus .....                                                                                                         | 139        |
|                  | Case I: High Cost of Control ( $r \rightarrow \infty$ ) .....                                                                          | 140        |
|                  | Case II: Low Cost of Control ( $r \rightarrow 0$ ) .....                                                                               | 140        |
| 3.7              | Linear Quadratic Trajectory Control .....                                                                                              | 143        |
| 3.8              | Frequency-Shaped LQ Control .....                                                                                                      | 148        |
| 3.9              | Minimum-Time Control of a Linear Time-Invariant System .....                                                                           | 154        |
|                  | Normality of Linear Systems .....                                                                                                      | 155        |
|                  | Existence and Uniqueness Theorems on Minimum-Time Control .....                                                                        | 156        |
|                  | Exercise Problems .....                                                                                                                | 168        |
| <b>Chapter 4</b> | <b>Control with Estimated States .....</b>                                                                                             | <b>179</b> |
| 4.1              | Open-Loop Observer .....                                                                                                               | 179        |
| 4.2              | Closed-Loop Observer .....                                                                                                             | 180        |
|                  | 4.2.1 Determination of Observer Gain Vector $L$ for a<br>Single-Output System .....                                                    | 181        |
|                  | 4.2.2 Determination of Observer Gain Matrix $L$ for a<br>Multiple-Output System .....                                                  | 182        |
|                  | 4.2.3 Locations of Observer Poles .....                                                                                                | 182        |
| 4.3              | Combined Observer-CONTROLLER .....                                                                                                     | 186        |
| 4.4              | Reduced-Order Observer .....                                                                                                           | 191        |
| 4.5              | Response of a Linear Continuous-Time System to White Noise .....                                                                       | 195        |
| 4.6              | Kalman Filter: Optimal State Estimation .....                                                                                          | 200        |
|                  | State Dynamics .....                                                                                                                   | 200        |
|                  | Measurement Equation .....                                                                                                             | 200        |
|                  | Observer Dynamics .....                                                                                                                | 201        |
| 4.7              | Stochastic Optimal Regulator in Steady State .....                                                                                     | 208        |
|                  | State Dynamics .....                                                                                                                   | 208        |
|                  | Objective Function .....                                                                                                               | 209        |
| 4.8              | Linear Quadratic Gaussian (LQG) Control .....                                                                                          | 215        |
|                  | State Dynamics .....                                                                                                                   | 216        |
|                  | Measurement Equation .....                                                                                                             | 216        |
|                  | Objective Function .....                                                                                                               | 216        |
|                  | Closed-Loop System Dynamics .....                                                                                                      | 218        |
|                  | Minimum Value of the Objective Function .....                                                                                          | 219        |
| 4.9              | Impact of Modeling Errors on Observer-Based Control .....                                                                              | 222        |
|                  | 4.9.1 Structured Parametric Uncertainties .....                                                                                        | 222        |
|                  | 4.9.2 Unmodeled Dynamics .....                                                                                                         | 224        |
|                  | Exercise Problems .....                                                                                                                | 228        |
| <b>Chapter 5</b> | <b>Robust Control: Fundamental Concepts and <math>H_2</math>, <math>H_\infty</math>, and<br/>    <math>\mu</math> Techniques .....</b> | <b>235</b> |
| 5.1              | Important Aspects of Singular Value Analysis .....                                                                                     | 235        |

|       |                                                                                                                    |     |
|-------|--------------------------------------------------------------------------------------------------------------------|-----|
| 5.1.1 | Significance of Minimum and Maximum Singular Values of the Transfer Function Matrix at a Frequency .....           | 235 |
| 5.1.2 | An Inequality for Robustness Test .....                                                                            | 237 |
| 5.2   | Robustness: Sensitivity and Complementary Sensitivity .....                                                        | 238 |
| 5.2.1 | Basic Definitions .....                                                                                            | 238 |
| 5.2.2 | Robustness to Structured Uncertainties .....                                                                       | 242 |
| 5.2.3 | Robustness to Unstructured Uncertainties .....                                                                     | 244 |
| 5.3.  | Robustness of LQR and Kalman Filter (KF) Feedback Loops .....                                                      | 252 |
| 5.3.1 | Liner Quadratic Regulator (LQR) Feedback Loop .....                                                                | 252 |
| 5.3.2 | Gain and Phase Margins of a Single-Input System .....                                                              | 254 |
| 5.3.3 | Gain and Phase Margins of a Multiinput System .....                                                                | 257 |
| 5.3.2 | Kalman Filter (KF) Loop .....                                                                                      | 260 |
| 5.4   | LQG/LTR Control .....                                                                                              | 263 |
| 5.4.1 | Lack of Guaranteed Robustness of LQG Control .....                                                                 | 263 |
| 5.4.2 | Loop Transfer Recovery (LTR) .....                                                                                 | 265 |
| 5.5   | $H_2$ And $H_\infty$ Norms .....                                                                                   | 269 |
| 5.5.1 | Definition of $L_2$ Norm .....                                                                                     | 269 |
| 5.5.2 | Definition of $H_2$ Norm .....                                                                                     | 269 |
| 5.5.3 | Significance of $H_2$ Norm .....                                                                                   | 270 |
| I.    | Connection with Unit Impulse Responses .....                                                                       | 270 |
| II.   | Connection with the Root Mean Square (RMS) Response .....                                                          | 271 |
| 5.5.4 | Definition of $H_\infty$ Norm .....                                                                                | 273 |
| 5.5.5 | Significance of $H_\infty$ Norm .....                                                                              | 273 |
| I.    | Connection with the Worst-Case Frequency Response .....                                                            | 273 |
| II.   | Connection with the Worst-Case Output or Input .....                                                               | 274 |
| III.  | Connection of the Bounded Real Lemma (5.5.28 to 5.5.30) with a Special Linear Quadratic Maximization Problem ..... | 275 |
| 5.5.6 | Computation of $H_2$ Norm .....                                                                                    | 278 |
| 5.5.7 | Computation of $H_\infty$ Norm .....                                                                               | 281 |
| 5.6   | $H_2$ Control .....                                                                                                | 283 |
| 5.6.1 | Full State Feedback $H_2$ Control (Figure 5.6.1) .....                                                             | 283 |
| 5.6.2 | Output Feedback $H_2$ Control (Figure 5.6.2) .....                                                                 | 286 |
| 5.7   | Well-Posedness, Internal Stability, and Small Gain Theorem .....                                                   | 288 |
| 5.7.1 | Well-Posedness and Internal Stability of a General Feedback System .....                                           | 288 |
|       | Well-Posedness of Feedback System .....                                                                            | 290 |
|       | Internal Stability of the System .....                                                                             | 290 |
| 5.7.2 | Small Gain Theorem .....                                                                                           | 292 |
| 5.7.3 | Analysis for Application of Small Gain Theorem .....                                                               | 297 |
| 5.8   | Formulation of Some Robust Control Problems with Unstructured Uncertainties .....                                  | 299 |
| 5.8.1 | Multiplicative Uncertainty .....                                                                                   | 299 |
| 5.8.2 | Additive Uncertainty .....                                                                                         | 301 |

|                  |                                                                                                                                |            |
|------------------|--------------------------------------------------------------------------------------------------------------------------------|------------|
| 5.9              | Formulation of Robust Control Problems with Structured Uncertainties.....                                                      | 304        |
| 5.9.1            | Linear Fractional Transformation (LFT).....                                                                                    | 304        |
|                  | Lower LFT .....                                                                                                                | 305        |
|                  | Upper LFT.....                                                                                                                 | 306        |
| 5.9.2            | Structured Parametric Uncertainties .....                                                                                      | 308        |
| 5.10             | $H_\infty$ Control.....                                                                                                        | 314        |
| 5.10.1           | Full State Feedback $H_\infty$ Control (Figure 5.10.1).....                                                                    | 314        |
| 5.10.2           | Full State Feedback $H_\infty$ Control under Disturbance Feedforward (Figure 5.10.3).....                                      | 320        |
| 5.10.3           | Guaranteed $H_\infty$ Norm via State Estimation .....                                                                          | 322        |
| 5.10.4           | Output Feedback $H_\infty$ Control.....                                                                                        | 324        |
|                  | Equivalence of Stability of Systems Shown in Figure 5.10.6 and Figure 5.10.7 .....                                             | 328        |
|                  | Controller Design .....                                                                                                        | 332        |
| 5.10.5           | Poles or Zeros on Imaginary Axis .....                                                                                         | 336        |
| 5.11             | Loop Shaping .....                                                                                                             | 337        |
| 5.11.1           | Trade-Off between Performance and Robustness via $H_\infty$ Control.....                                                       | 337        |
| 5.11.2           | Fundamental Constraint: Bode's Sensitivity Integrals .....                                                                     | 345        |
|                  | SISO System .....                                                                                                              | 345        |
| 5.12             | Controller Based on $\mu$ Analysis.....                                                                                        | 351        |
| 5.12.1           | Definitions and Properties of Structured Singular Values ( $\mu$ ) .....                                                       | 351        |
| 5.12.2           | Robustness Analysis via $\mu$ .....                                                                                            | 353        |
| 5.12.3           | Robust Performance via $\mu$ Analysis .....                                                                                    | 354        |
| 5.12.4           | $\mu$ Synthesis: $D$ - $K$ Iteration .....                                                                                     | 366        |
|                  | Exercise Problems .....                                                                                                        | 367        |
| <b>Chapter 6</b> | <b>Robust Control: Sliding Mode Methods.....</b>                                                                               | <b>377</b> |
| 6.1              | Basic Concepts of Sliding Modes .....                                                                                          | 377        |
| 6.2              | Sliding Mode Control of a Linear System with Full State Feedback.....                                                          | 379        |
| 6.2.1            | Computation of Sliding Hyperplane Matrix $G$ .....                                                                             | 380        |
| 6.2.2            | Optimal Sliding Mode (OS) Controller.....                                                                                      | 383        |
| 6.3              | Sliding Mode Control of an Uncertain Linear System with Full State Feedback: Blending $H_\infty$ and Sliding Mode Methods..... | 386        |
| 6.3.1            | Impact of Uncertainties on System Dynamics in Sliding Modes .....                                                              | 387        |
| 6.3.2            | Computation of Sliding Hyperplane Matrix via $H_\infty$ Control Method .....                                                   | 388        |
| 6.4              | Sliding Mode Control of a Linear System with Estimated States.....                                                             | 394        |
| 6.5              | Optimal Sliding Mode Gaussian (OSG) Control .....                                                                              | 396        |
|                  | Exercise Problems .....                                                                                                        | 402        |

|                         |            |
|-------------------------|------------|
| <b>References .....</b> | <b>405</b> |
|-------------------------|------------|

|                                                                                                           |            |
|-----------------------------------------------------------------------------------------------------------|------------|
| <b>Appendix A: Linear Algebraic Equations, Eigenvalues/Eigenvectors, and Matrix Inversion Lemma .....</b> | <b>409</b> |
| A.1 System of Linear Algebraic Equations .....                                                            | 409        |
| Definitions .....                                                                                         | 409        |
| Linear Independence of Vectors .....                                                                      | 409        |
| Range or Column Space .....                                                                               | 409        |
| Rank of a Matrix .....                                                                                    | 409        |
| Null Space .....                                                                                          | 410        |
| A Systematic Test for Linear Independence of Vectors .....                                                | 414        |
| A.2 Eigenvalues and Eigenvectors .....                                                                    | 415        |
| A.3 Matrix Inversion Lemma .....                                                                          | 415        |

|                                                                                                                 |            |
|-----------------------------------------------------------------------------------------------------------------|------------|
| <b>Appendix B: Quadratic Functions, Important Derivatives, Fourier Integrals, and Parseval's Relation .....</b> | <b>417</b> |
| B.1 Quadratic Functions .....                                                                                   | 417        |
| B.2 Derivative of a Quadratic Function .....                                                                    | 419        |
| B.3 Derivative of a Linear Function .....                                                                       | 421        |
| B.4 Fourier Integrals and Parseval's Theorem .....                                                              | 422        |
| B.4.1 Scalar Signal .....                                                                                       | 422        |
| Scalar Parseval Relation .....                                                                                  | 424        |
| B.4.2 Vector Signal .....                                                                                       | 424        |
| Multivariable Parseval Relation .....                                                                           | 425        |

|                                                                                                 |            |
|-------------------------------------------------------------------------------------------------|------------|
| <b>Appendix C: Norms, Singular Values, Supremum, and Infimum .....</b>                          | <b>427</b> |
| C.1 Vector Norms .....                                                                          | 427        |
| C.2 Matrix Norms .....                                                                          | 427        |
| C.3 Singular Values of a Matrix .....                                                           | 429        |
| C.4 Singular Value Decomposition (SVD) .....                                                    | 429        |
| Computation of $U$ and $V$ .....                                                                | 430        |
| Significance of Singular Vectors Corresponding to the Maximum and Minimum Singular Values ..... | 431        |
| C.5 Properties of Singular Values .....                                                         | 432        |
| C.6 Supremum and Infimum .....                                                                  | 432        |

|                                                          |            |
|----------------------------------------------------------|------------|
| <b>Appendix D: Stochastic Processes .....</b>            | <b>435</b> |
| D.1 Stationary Stochastic Process .....                  | 435        |
| Autocorrelation Function .....                           | 436        |
| Autocovariance Function .....                            | 437        |
| Crosscorrelation Function .....                          | 437        |
| Crosscovariance Function .....                           | 437        |
| Uncorrelated Stochastic Processes .....                  | 438        |
| D.2 Power Spectrum or Power Spectral Density (PSD) ..... | 438        |

|                                                                                                                                          |                                                                                                            |         |
|------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------|---------|
| D.3                                                                                                                                      | White Noise: A Special Stationary Stochastic Process.....                                                  | 439     |
|                                                                                                                                          | Generation of Approximate Gaussian White Noise in MATLAB .....                                             | 439     |
| D.4                                                                                                                                      | Response of a SISO Linear and Time-Invariant System Subjected,<br>to a Stationary Stochastic Process ..... | 440     |
| D.5                                                                                                                                      | Vector Stationary Stochastic Processes .....                                                               | 440     |
| <br><b>Appendix E: Optimization of a Scalar Function with Constraints<br/>in the Form of a Symmetric Real Matrix Equal to Zero .....</b> |                                                                                                            | <br>443 |
| <br><b>Appendix F: A Flexible Tetrahedral Truss Structure.....</b>                                                                       |                                                                                                            | <br>447 |
| <br><b>Appendix G: Space Shuttle Dynamics during Reentry.....</b>                                                                        |                                                                                                            | <br>451 |
|                                                                                                                                          | State Space Model .....                                                                                    | 453     |
| <br><b>Index.....</b>                                                                                                                    |                                                                                                            | <br>457 |