

Finite Difference Methods for Ordinary and Partial Differential Equations

Steady-State and Time-Dependent Problems

Randall J. LeVeque



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