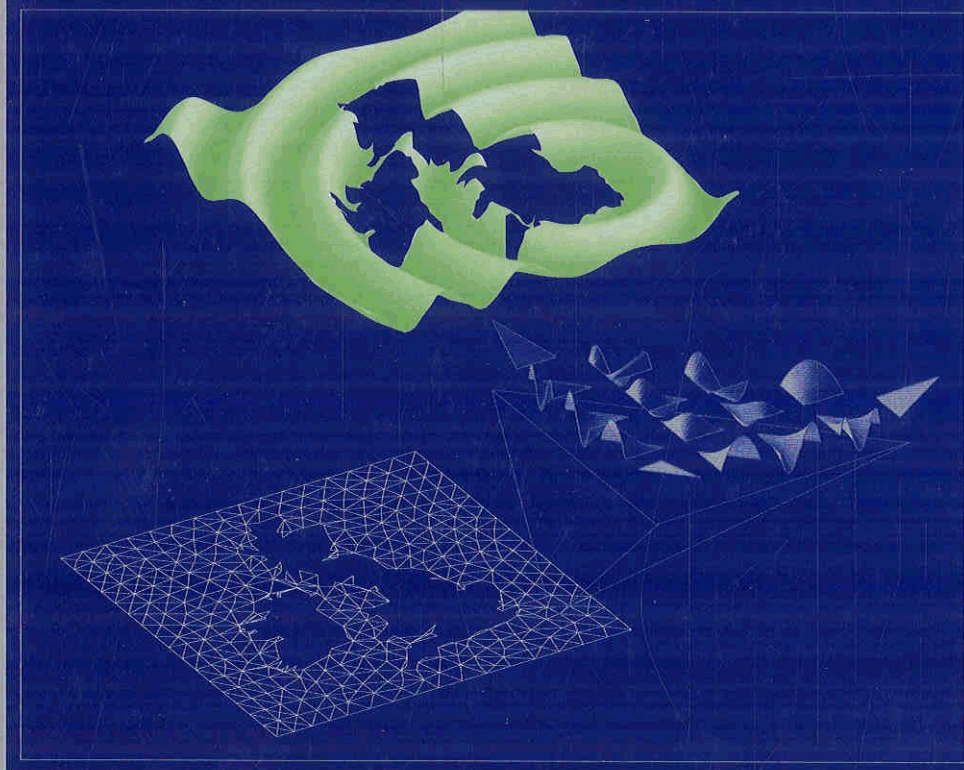


NUMERICAL MATHEMATICS
AND SCIENTIFIC COMPUTATION

Spectral/hp Element Methods for Computational Fluid Dynamics

SECOND EDITION

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