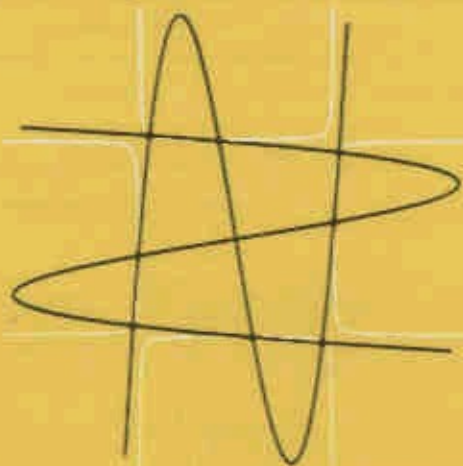



# Graduate Texts in Mathematics

David Eisenbud

## The Geometry of Syzygies

A Second Course in Commutative  
Algebra and Algebraic Geometry



 Springer

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