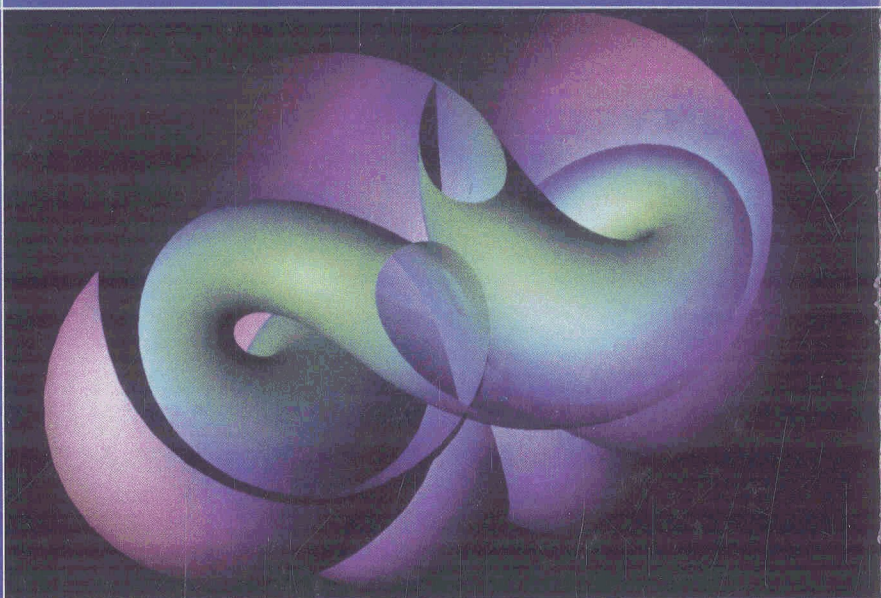


GRADUATE STUDENT SERIES IN PHYSICS



GEOMETRY, TOPOLOGY AND PHYSICS

Second Edition

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