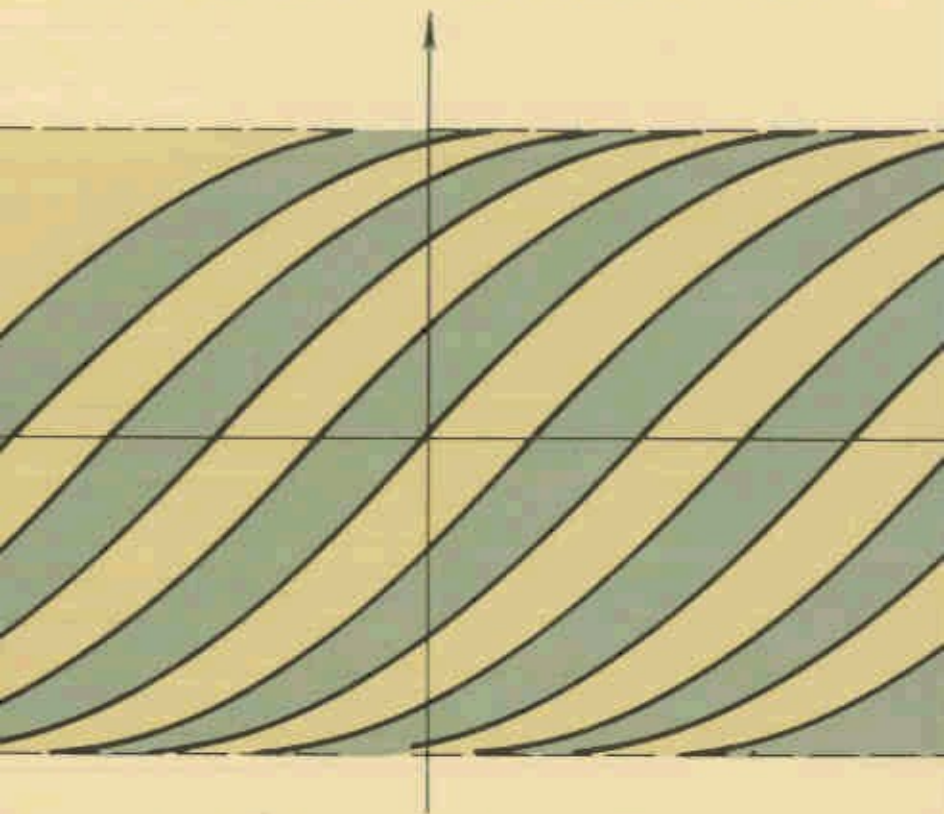


INTRODUCTION TO
THE
CALCULUS
OF
VARIATIONS



Hans Sagan

CONTENTS

PREFACE	vii
ACKNOWLEDGMENTS	xi
CHAPTER 1 EXTREME VALUES OF FUNCTIONALS	1
1.1 INTRODUCTION	1
1.2 FUNCTIONALS	9
1.3 NECESSARY CONDITIONS FOR RELATIVE EXTREME VALUES OF REAL-VALUED FUNCTIONS OF ONE REAL VARIABLE	11
1.4 NORMED LINEAR SPACES	16
1.5 THE GÂTEAUX VARIATION OF A FUNCTIONAL	25
1.6 THE SPACE OF ADMISSIBLE VARIATIONS	29
1.7 FIRST NECESSARY CONDITION FOR A RELATIVE MINIMUM OF A FUNCTIONAL	32
1.8 THE SECOND GÂTEAUX VARIATION AND A SECOND NECESSARY CONDITION FOR A RELATIVE MINIMUM OF A FUNCTIONAL	35
BRIEF SUMMARY	40
APPENDIX	41
A1.9 RELATIVE EXTREME VALUES OF REAL-VALUED FUNCTIONS OF n REAL VARIABLES	41
CHAPTER 2 THE THEORY OF THE FIRST VARIATION	44
2.1 WEAK AND STRONG RELATIVE EXTREME VALUES	45
2.2 FIRST NECESSARY CONDITION FOR THE SIMPLEST VARIATIONAL PROBLEM	48
2.3 THE EULER-LAGRANGE EQUATION	49
2.4 LAGRANGE'S METHOD	54
2.5 DISCUSSION OF THE EULER-LAGRANGE EQUATION	56
2.6 THE PROBLEM OF MINIMAL SURFACES OF REVOLUTION	62
2.7 NATURAL BOUNDARY CONDITIONS	66
2.8 TRANSVERSALITY CONDITIONS	68
2.9 BROKEN EXTREMALS AND THE WEIERSTRASS-ERDMANN CORNER CONDITIONS	75
2.10 SMOOTHING OF CORNERS	82
2.11 GENERALIZATION TO MORE THAN ONE UNKNOWN FUNCTION	88
2.12 THE EULER-LAGRANGE EQUATIONS IN CANONICAL FORM	95
BRIEF SUMMARY	100
APPENDIX	102
A2.13 THE PROBLEM IN TWO UNKNOWN FUNCTIONS WITH VARIABLE ENDPOINTS	102
A2.14 INVARIANCE OF THE EULER-LAGRANGE EQUATIONS	107
	xiii

A2.15 HAMILTON'S PRINCIPLE OF STATIONARY ACTION	116
A2.16 NOETHER'S INTEGRATION OF THE EULER-LAGRANGE EQUATION-CONSERVATION LAWS IN MECHANICS	118
A2.17 GENERALIZATION TO MORE THAN ONE INDEPENDENT VARIABLE	124
CHAPTER 3 THEORY OF FIELDS AND SUFFICIENT CONDITIONS FOR A STRONG RELATIVE EXTREMUM	131
3.1 FIELDS	132
3.2 HILBERT'S INVARIANT INTEGRAL	137
3.3 TRANSFORMATION OF THE TOTAL VARIATION	140
3.4 AN EXAMPLE OF A STRONG MINIMUM	142
3.5 FIELD CONSTRUCTION AND THE JACOBI EQUATION	147
3.6 THE ZEROS OF THE SOLUTIONS OF THE JACOBI EQUATION-CONJUGATE POINTS	150
3.7 CONJUGATE POINTS AND FIELD EXISTENCE	154
3.8 A SUFFICIENT CONDITION FOR A WEAK MINIMUM	158
3.9 A NECESSARY CONDITION FOR A STRONG RELATIVE MINIMUM	161
3.10 A SUFFICIENT CONDITION FOR THE PROBLEM IN n UNKNOWN FUNCTIONS	167
BRIEF SUMMARY	172
APPENDIX	173
A3.11 SUFFICIENT CONDITIONS FOR THE VARIABLE-ENDPOINT PROBLEM	173
A3.12 EXISTENCE OF A TRANSVERSAL FIELD	180
A3.13 FOCAL POINTS IN TRANSVERSAL FIELDS	183
A3.14 FIELD, INVARIANT INTEGRAL, AND EXCESS FUNCTION OF THE PROBLEM IN TWO INDEPENDENT VARIABLES	192
CHAPTER 4 THE HOMOGENEOUS PROBLEM	197
4.1 PARAMETER INVARIANCE OF INTEGRAL	197
4.2 PROPERTIES OF HOMOGENEOUS FUNCTIONS	203
4.3 WEAK AND STRONG RELATIVE EXTREMA	205
4.4 THE EULER-LAGRANGE EQUATIONS FOR THE HOMOGENEOUS PROBLEM	211
4.5 DISCUSSION OF THE EULER-LAGRANGE EQUATIONS	215
4.6 TRANSVERSALITY CONDITION	222
4.7 CARATHEODORY'S INDICATRIX	228
4.8 INTEGRALS OF THE EULER-LAGRANGE EQUATIONS	236
4.9 FIELD AND EXCESS FUNCTION	239
4.10 STRONG AND WEAK EXTREMA	243
BRIEF SUMMARY	251

CHAPTER 5 THE HAMILTON-JACOBI THEORY AND THE MINIMUM PRINCIPLE OF PONTRYAGIN	253
5.1 A FUNDAMENTAL LEMMA OF CARATHÉODORY	253
5.2 DYNAMIC PROGRAMMING	260
5.3 THE HAMILTON-JACOBI EQUATION	264
5.4 SOLUTION OF THE HAMILTON-JACOBI EQUATION—JACOBI'S THEOREM	269
5.5 THE HAMILTON-JACOBI EQUATION AND FIELD EXISTENCE	274
5.6 A GENERAL MINIMUM-INTEGRAL CONTROL PROBLEM	277
5.7 THE MINIMUM PRINCIPLE OF PONTRYAGIN	283
BRIEF SUMMARY	292
APPENDIX	294
A5.8 THE TIME-OPTIMAL CONTROL PROBLEM	294
A5.9 A NONAUTONOMOUS TERMINAL CONTROL PROBLEM OF PREDETERMINED DURATION	301
A5.10 THE MINIMUM PRINCIPLE AS A SUFFICIENT CONDITION FOR LINEAR CONTROL PROBLEMS OF FIXED DURATION	304
A5.11 BANG-BANG CONTROLS	310
A5.12 A PROBLEM OF LAGRANGE AS AN OPTIMAL CONTROL PROBLEM	316
CHAPTER 6 THE PROBLEM OF LAGRANGE AND THE ISOPERIMETRIC PROBLEM	325
6.1 VARIATIONAL PROBLEMS WITH CONSTRAINTS	325
6.2 THE PROBLEM OF MAYER AND A FUNDAMENTAL THEOREM ON UNDERDETERMINED SYSTEMS	329
6.3 THE LAGRANGE MULTIPLIER RULE	332
6.4 DISCUSSION OF THE LAGRANGE MULTIPLIER RULE	336
6.5 THE ISOPERIMETRIC PROBLEM	339
6.6 DISCUSSION OF THE ISOPERIMETRIC PROBLEM	343
6.7 PROOF OF THE FUNDAMENTAL THEOREM ON UNDERDETERMINED SYSTEMS	346
6.8 THE MAYER PROBLEM WITH A VARIABLE ENDPOINT	357
6.9 TRANSVERSALITY CONDITIONS FOR THE LAGRANGE PROBLEM WITH A VARIABLE ENDPOINT	364
6.10 A SUFFICIENT CONDITION FOR THE LAGRANGE PROBLEM	371
BRIEF SUMMARY	380
APPENDIX	383
A6.11 ON THE AUGMENTATION OF A MATRIX	383
A6.12 A LAGRANGE PROBLEM WITH FINITE CONSTRAINTS	386

CHAPTER 7 THE THEORY OF THE SECOND VARIATION	390
7.1 NECESSARY AND SUFFICIENT CONDITIONS FOR A WEAK MINIMUM	390
7.2 LEGENDRE'S NECESSARY CONDITION	393
7.3 BLISS' SECONDARY VARIATIONAL PROBLEM AND JACOBI'S NECESSARY CONDITION	396
7.4 LEGENDRE'S TRANSFORMATION OF THE SECOND VARIATION	399
7.5 A SUFFICIENT CONDITION FOR A WEAK RELATIVE MINIMUM	405
7.6 SCHEMATIC REVIEW OF THE SIMPLEST VARIATIONAL PROBLEM	408
7.7 THE SECOND VARIATION OF FUNCTIONALS OF n VARIABLES	411
7.8 THE STRENGTHENED LEGENDRE CONDITION	415
7.9 CONJUGATE POINTS AND JACOBI'S NECESSARY CONDITION	417
BRIEF SUMMARY	420
APPENDIX	422
A7.10 THE LEGENDRE CONDITION FOR THE HOMOGENEOUS PROBLEM	422
A7.11 THE JACOBI CONDITION FOR THE HOMOGENEOUS PROBLEM	428
BIBLIOGRAPHY	434
INDEX	439