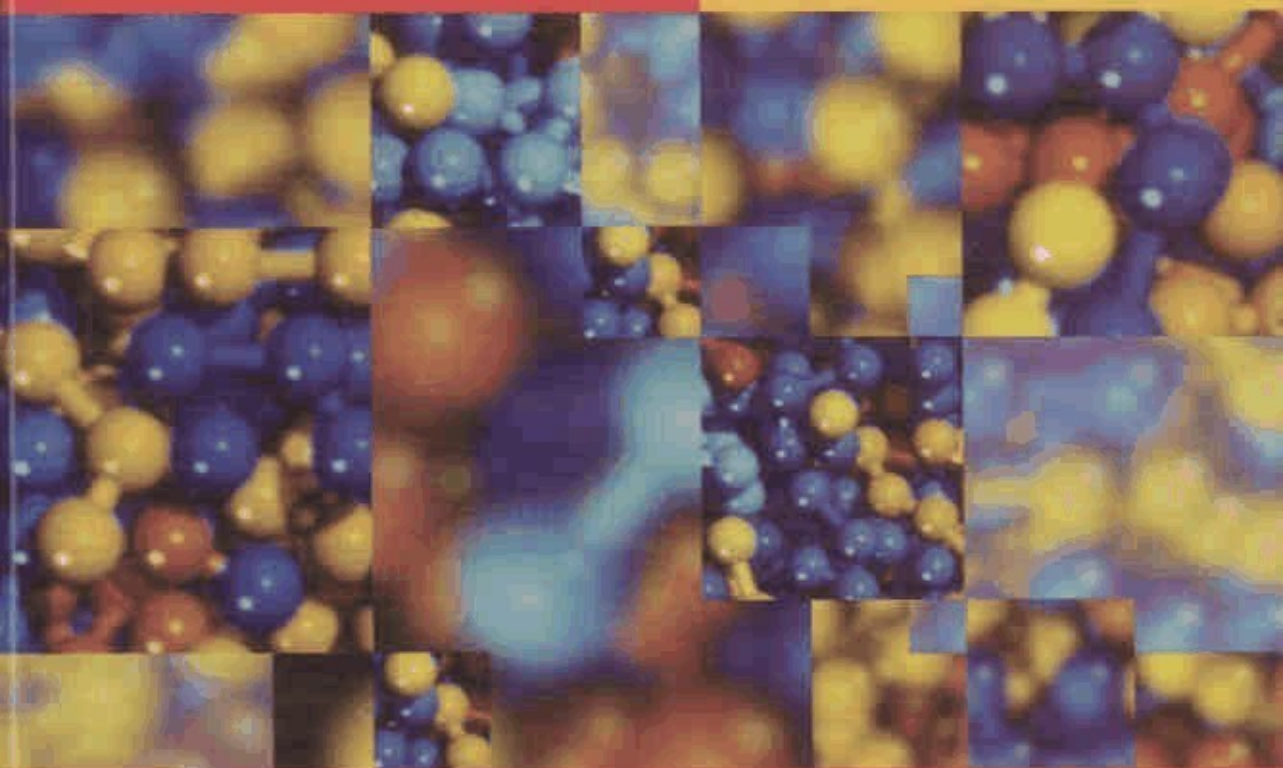


SECOND EDITION

A Guide to
Monte Carlo Simulations in
Statistical Physics

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CAMBRIDGE

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