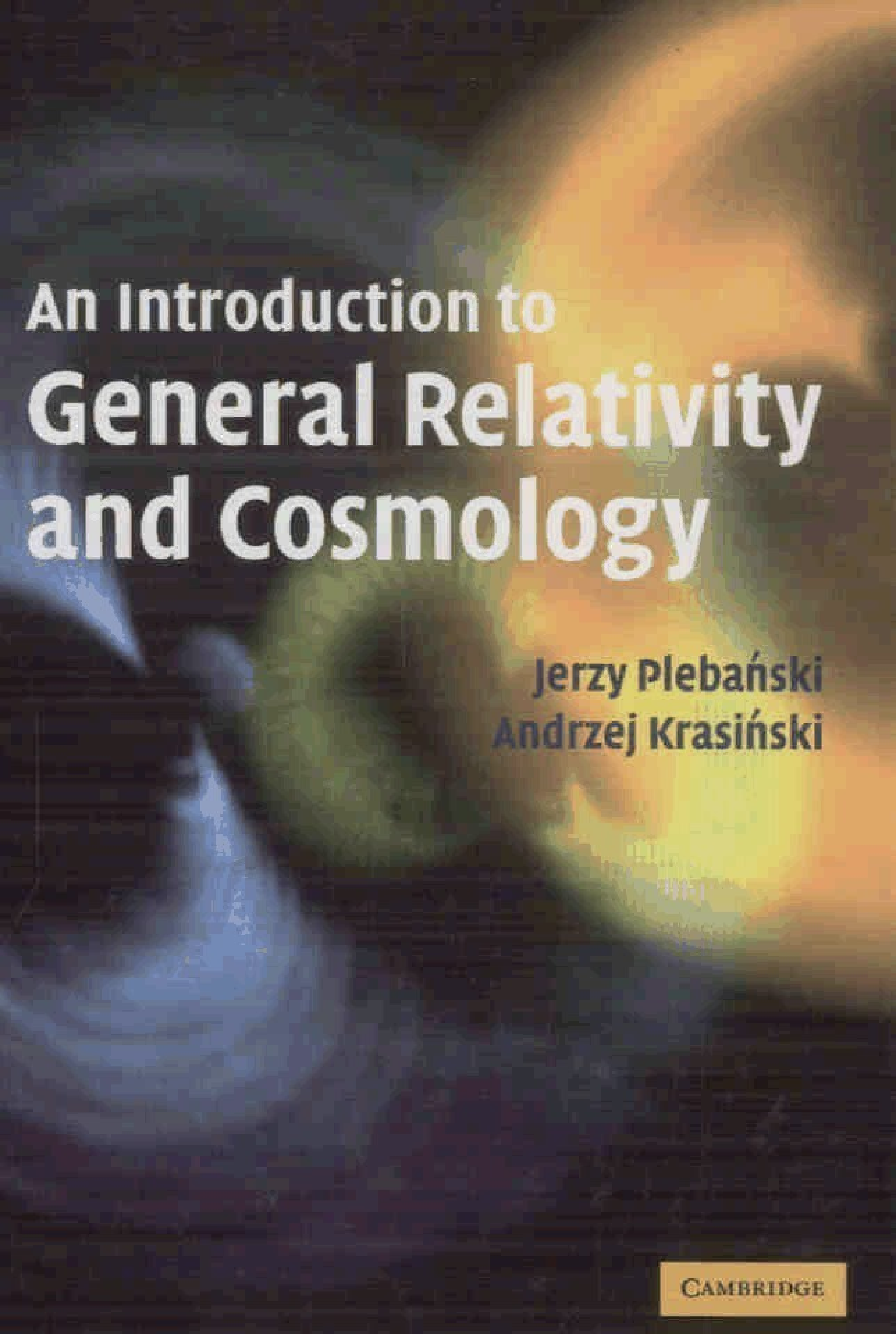

An Introduction to General Relativity and Cosmology

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CAMBRIDGE

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