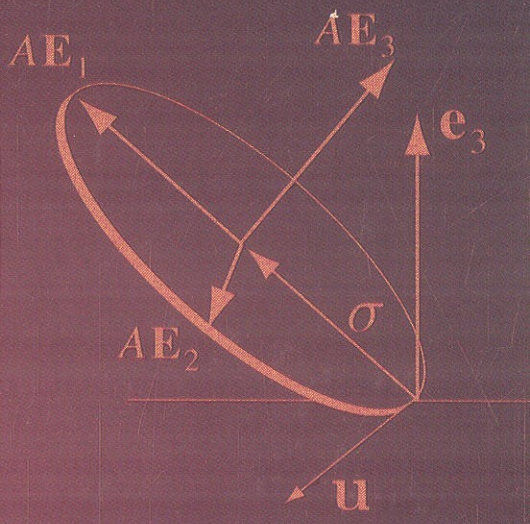

GEOMETRIC MECHANICS

Part II: Rotating, Translating and Rolling

DARRYL D HOLM

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