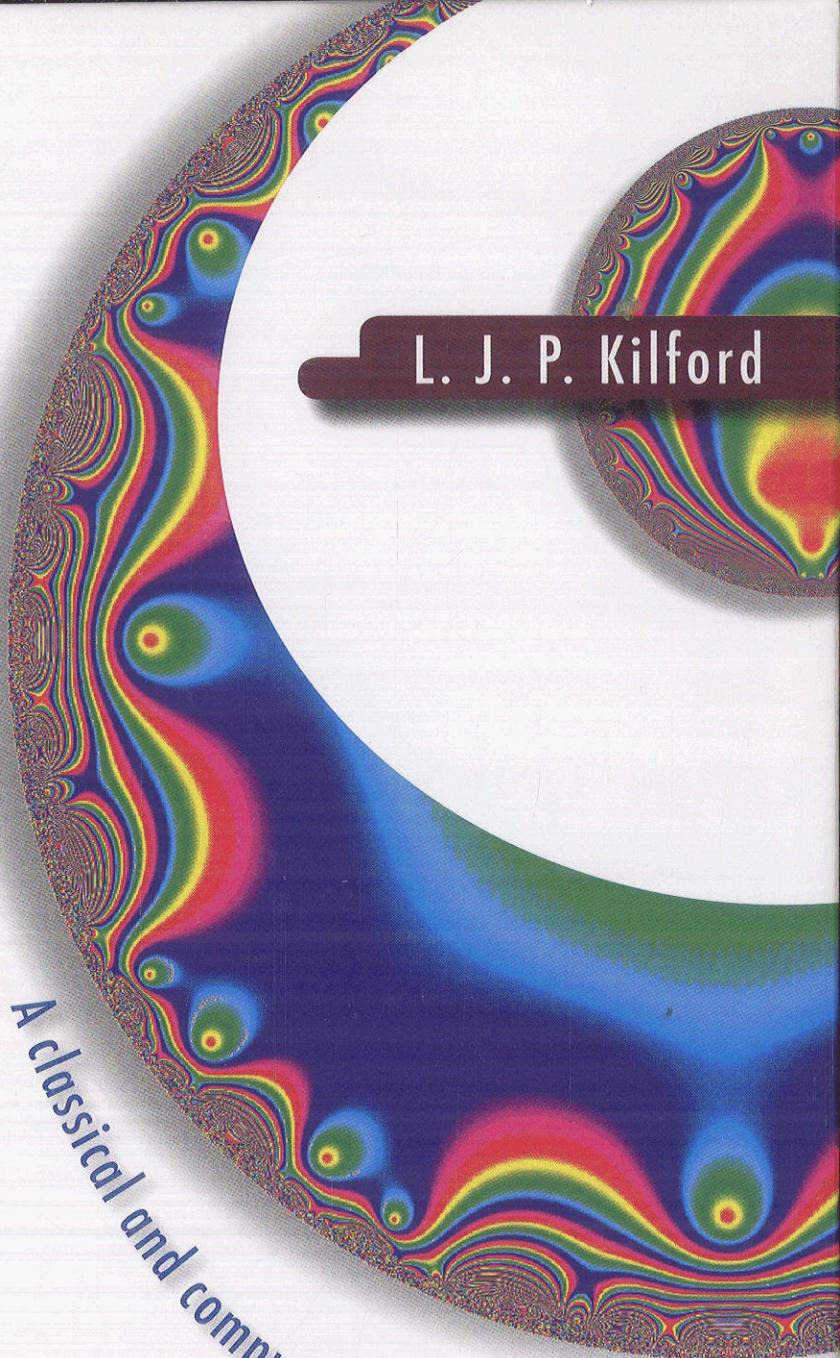


Modular Forms



L. J. P. Kilford

A classical and computational introduction

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