

# Linear Models for Unbalanced Data

SHAYLE R. SEARLE

WILEY SERIES IN PROBABILITY AND STATISTICS

# CONTENTS

---

<b>1. An Up-Dated Viewpoint: Cell Means Models</b>	<b>1</b>
1.1. Statistics and computers	1
1.2. Balanced and unbalanced data	3
<i>a. Factors, levels, effects and cells, 3</i>	
<i>b. Balanced data, 5</i>	
<i>c. Special cases of unbalanced data, 5</i>	
<i>d. Unbalanced data, 7</i>	
<i>e. A summary, 8</i>	
1.3. Cell means models	10
1.4. Statistical computing packages	13
1.5. Hypothesis testing	14
 <b>2. Basic Results for Cell Means Models: The 1-Way Classification</b>	 <b>17</b>
2.1. An example	17
2.2. The model and model equations	18
<i>a. The model equations, 18</i>	
<i>b. The model, 19</i>	
2.3. Estimation	21
2.4. Expected values and sampling variances	23
2.5. Estimating $E(y_{ij})$ and future observations	24
2.6. Estimating the error variance	25
2.7. Reductions in sums of squares: the $R(\cdot)$ notation	26
2.8. Partitioning the total sum of squares	29
<i>a. For the analysis of variance, 29</i>	
<i>b. The coefficient of determination, 31</i>	

<b>2. Cell Means (Continued)</b>	
2.9. The best linear unbiased estimator (BLUE)	32
<i>a. Linear functions, 32</i>	
<i>b. Definition of BLUE, 32</i>	
<i>c. Verification: <math>\hat{\mu}_i</math> is the BLUE of <math>\mu_i</math>, 32</i>	
<i>d. Derivation of <math>\hat{\mu}_i</math> as the BLUE, 34</i>	
<i>e. A linear function of BLUEs is a BLUE, 34</i>	
2.10. Normality assumptions	35
<i>a. The <math>y_{ij}</math>s are normal, 35</i>	
<i>b. The BLUE, <math>\hat{\mu}_i</math>, is normal, 35</i>	
<i>c. <math>\hat{\mu}_i</math> and <math>\hat{\sigma}^2</math> are independent, 36</i>	
<i>d. <math>SSR_m/\sigma^2</math> is distributed as <math>\chi^2_{a-1}</math>, 36</i>	
<i>e. <math>SSE/\sigma^2</math> is distributed as <math>\chi^2_{N-a}</math>, 37</i>	
<i>f. <math>SSR_m</math> and <math>SSE</math> are independent, 37</i>	
<i>g. F-statistics, 37</i>	
2.11. The analysis of variance table	38
<i>a. A summary of arithmetic, 38</i>	
<i>b. Tests of hypotheses, 39</i>	
2.12. Linear combinations of cell means	40
<i>a. Confidence intervals, 40</i>	
<i>b. Hypothesis tests, 42</i>	
<i>-i. One-part hypotheses, 42</i>	
<i>-ii. Two-part hypotheses, 43</i>	
<i>c. Fitting the mean, 45</i>	
<i>d. Other forms of a grand mean, 46</i>	
<i>e. Contrasts, 47</i>	
<i>f. Orthogonal contrasts for balanced data, 47</i>	
<i>g. Orthogonal contrasts for unbalanced data, 50</i>	
2.13. The overparameterized model	53
2.14. Appendix	54
<i>a. Proof that <math>SSR_m/\sigma^2 \sim \chi^2_{a-1}</math> (Section 2.10d), 54</i>	
<i>b. Properties of <math>\hat{\delta}_i</math> (Section 2.12g), 57</i>	
2.15. Summary	58
<i>a. Model and estimation, 58</i>	
<i>b. Sums of squares and F-statistics, 58</i>	
2.16. Exercises	59
<b>3. Nested Classifications</b>	<b>61</b>
3.1. Notation	61
3.2. The model	63
3.3. Estimation	63
3.4. Confidence intervals, and one-part hypotheses	65

3.5.	Analysis of variance	65
a.	<i>For the cell means,</i>	65
b.	<i>Cell means within levels of the primary factor,</i>	67
c.	<i>Means for the levels of the primary factor,</i>	68
d.	<i>An overall mean,</i>	70
e.	<i>Summary,</i>	70
f.	<i>Using the hypotheses in the analysis of variance,</i>	71
3.6.	More than two factors	73
a.	<i>Models,</i>	73
b.	<i>Sums of squares,</i>	74
c.	<i>Hypotheses tested by analysis of variance sums of squares in the fixed effects model,</i>	75
3.7.	Exercises	77
4.	<b>The 2-Way Crossed Classification with All-Cells-Filled Data: Cell Means Models</b>	<b>78</b>
4.1.	Notation	78
4.2.	The model	81
4.3.	Estimation	81
4.4.	Confidence intervals, and one-part hypotheses	82
4.5.	Analysis of variance	84
a.	<i>A standard analysis,</i>	84
b.	<i>An alternative approach,</i>	85
4.6.	Unweighted means	86
a.	<i>Definitions,</i>	87
b.	<i>Hypothesis: equality of row means,</i>	88
c.	<i>Numerator sums of squares,</i>	88
d.	<i>A general hypothesis: row means equal,</i>	89
e.	<i>A similar result for columns,</i>	91
f.	<i>A caution,</i>	91
g.	<i>Least squares means,</i>	91
4.7.	Weighted means	92
a.	<i>Weighting by cell frequencies,</i>	92
b.	<i>Evaluation,</i>	93
c.	<i>A general weighting,</i>	94
d.	<i>Similar results for columns,</i>	94
4.8.	Interactions	95
a.	<i>A meaning of interaction,</i>	95
b.	<i>A measure of interaction,</i>	96
c.	<i>The number of interactions in a grid,</i>	97
d.	<i>A consequence of interactions,</i>	98

<b>4. The 2-Way Model: All-Cells-Filled (Continued)</b>	
4.9. Testing a hypothesis of no interactions	99
a. <i>A no-interaction model, 100</i>	
b. <i>Estimation in the no-interaction model, 101</i>	
c. <i>Reduction in sum of squares, 103</i>	
d. <i>The F-statistic, 104</i>	
e. <i>Consequences of testing for interactions, 105</i>	
-i. <i>Estimating means, 105</i>	
-ii. <i>Estimating the residual variance, 106</i>	
-iii. <i>Other F-statistics, 107</i>	
f. <i>Partitioning the total sum of squares, 109</i>	
g. <i>Analysis of variance, 111</i>	
h. <i>Hypotheses in the analysis of variance table, 113</i>	
i. <i>Balanced data, 116</i>	
j. <i>Computer output, 117</i>	
4.10. The no-interaction model	117
a. <i>Estimating means, 118</i>	
b. <i>F-statistics, 120</i>	
c. <i>Analysis of variance, 121</i>	
d. <i>Hypotheses in the analysis of variance table, 123</i>	
4.11. Appendix	124
4.12. Summary	127
a. <i>Model and estimation, 127</i>	
b. <i>Means of cell means, 127</i>	
c. <i>Interactions, 127</i>	
d. <i>The no-interaction model, 128</i>	
e. <i>Analysis of variance for the with-interaction model, 128</i>	
f. <i>Using the no-interaction model, 128</i>	
4.13. Exercises	129
<b>5. The 2-Way Classification with Some-Cells-Empty Data: Cell Means Models</b>	<b>132</b>
5.1. Preliminaries	132
a. <i>Model, 132</i>	
b. <i>Estimation, 132</i>	
c. <i>Linear functions of cell means, 133</i>	
d. <i>Estimating the residual variance, 134</i>	
e. <i>Confidence intervals, 134</i>	
f. <i>Example 1, 134</i>	
g. <i>Analysis of variance, 135</i>	
5.2. Estimability: an introduction	136
a. <i>Row means of cell means, 137</i>	
b. <i>Interactions, 138</i>	

5.3.	Connected data	139
	a. <i>Basic ideas</i> , 139	
	b. <i>A geometric algorithm</i> , 140	
	c. <i>Separation of disconnected data</i> , 141	
	d. <i>Estimability of contrasts, and of all cell means</i> , 143	
	e. <i>Connectedness for several factors</i> , 144	
5.4.	Testing for interactions	145
	a. <i>Fitting the model</i> , 145	
	b. <i>The reduction in sum of squares for the no-interaction model</i> , 148	
	c. <i>Testing the hypothesis of no-interaction</i> , 148	
	d. <i>What hypothesis?</i> , 149	
	e. <i>Specifying the hypothesis corresponding to <math>\mathcal{R}(\mu_{ij} \mu_i, \tau_j)</math></i> , 151	
	f. <i>Using the no-interaction model</i> , 153	
5.5.	Analysis of variance	154
	a. <i>Weighted squares of means analysis</i> , 155	
	b. <i>The all-cells-filled analysis of Table 4.8</i> , 155	
	c. <i>Disconnected data</i> , 157	
5.6.	Subset analyses for with-interaction models	158
	a. <i>Difficulties</i> , 158	
	b. <i>What is of interest?</i> , 158	
	c. <i>The investigator's role</i> , 158	
	d. <i>A procedure</i> , 159	
	e. <i>Subset analyses</i> , 160	
	f. <i>Interactions</i> , 160	
	g. <i>Examples</i> , 161	
	h. <i>Difficulties with subsets</i> , 163	
	i. <i>The investigator</i> , 164	
5.7.	Summary	165
	a. <i>Model and estimation</i> , 165	
	b. <i>No-interaction model</i> , 165	
	c. <i>Testing for no interactions</i> , 166	
	d. <i>Using the no-interaction model</i> , 166	
5.8.	Exercises	166
6.	<b>Models with Covariables (Analysis of Covariance): The 1-Way Classification</b>	<b>169</b>
6.1.	The single slope model	171
	a. <i>Model</i> , 171	
	b. <i>Estimation of means</i> , 172	
	-i. <i>Estimation and unbiasedness</i> , 172	
	-ii. <i>Sampling variances and covariances</i> , 173	
	c. <i>Estimating residual variance</i> , 175	

<b>6. Covariables: The 1-Way Model (Continued)</b>	
<i>d. Between- and within-classes sums of squares and products, 176</i>	
<i>e. Partitioning the total sum of squares, 178</i>	
<i>f. The equal-class-effects sub-model: a single line, 181</i>	
-i. <i>Fitting the single-line model, 183</i>	
-ii. <i>Partitioning SST, 183</i>	
<i>g. Establishing tests of hypotheses, 183</i>	
<i>h. Two examples of <math>\chi^2</math>-variables, 185</i>	
<i>i. Analysis of variance, 187</i>	
<i>j. Tests of intercepts, 188</i>	
6.2. The multiple slopes model: intra-class regression	190
<i>a. The model, 190</i>	
<i>b. Estimation of means, 191</i>	
-i. <i>Estimation and unbiasedness, 191</i>	
-ii. <i>Sampling variances and covariances, 192</i>	
<i>c. Estimating residual variance, 192</i>	
<i>d. Partitioning the total sum of squares, 194</i>	
<i>e. The equal-class-effects sub-model: a pencil of lines, 198</i>	
-i. <i>Fitting the model, 198</i>	
-ii. <i>Partitioning SST, 200</i>	
-iii. <i>Partitioning <math>R(\mu, b_i)</math>, 200</i>	
<i>f. Analysis of variance, 203</i>	
<i>g. Tests of intercepts, 205</i>	
6.3. Appendix	209
6.4. Summary	209
<i>a. Single slope model, 210</i>	
<i>b. Multiple slopes model, 210</i>	
6.5. Exercises	210
<b>7. Matrix Algebra and Quadratic Forms (A Prelude to Chapter 8)</b>	<b>212</b>
7.1. Matrix algebra	212
<i>a. Notation, 212</i>	
<i>b. The rank and trace of a matrix, 213</i>	
<i>c. Eigenvalues and eigenvectors, 214</i>	
<i>d. Idempotent matrices, 214</i>	
<i>e. Summing vectors and <math>\mathbf{J}</math> matrices, 214</i>	
<i>f. Positive definite and allied matrices, 215</i>	
<i>g. Dispersion matrices, 215</i>	
<i>h. Generalized inverse matrices, 216</i>	
<i>i. Generalized inverses of <math>\mathbf{X}'\mathbf{X}</math>, 217</i>	
<i>j. Calculating a generalized inverse, 219</i>	
7.2. Solving linear equations	220
<i>a. Solutions, 220</i>	
<i>b. An invariance property, 221</i>	

7.3.	Partitioning $X'X$	224
	a. Generalized inverses, 224	
	b. Rank, 225	
7.4.	Non-central $\chi^2$ and $f$	226
	a. Normal distributions, 226	
	b. $\chi^2$ -distributions, 228	
	c. $F$ -distributions, 228	
7.5.	Quadratic forms	230
	a. Mean and variance, 231	
	b. The $\chi^2$ -distribution of a quadratic form, 232	
	c. Independence of two quadratic forms, 232	
7.6.	Hypothesis testing	233
	a. A general procedure, 233	
	b. Alternative forms of a hypothesis, 234	
	-i. Scrutinizing $\theta'A\theta$ , 234	
	-ii. Using expected values, 234	
	-iii. Too many statements, 235	
	-iv. Using eigenvectors of $A$ , 235	
7.7.	Exercises	238

## 8. A General Linear Model 241

8.1.	The model	243
8.2.	Normal equations and their solutions	246
	a. The general case, 246	
	b. Examples, 247	
	c. The 1-way classification, 253	
8.3.	Using a solution to the normal equations	254
	a. Mean, and dispersion matrix, of $\beta^0$ , 254	
	b. Estimating $E(y)$ , 255	
	c. Residual sum of squares, 256	
	d. Estimating residual error variance, 258	
8.4.	Partitioning the total sum of squares	259
	a. Reduction in sum of squares, 259	
	b. Sum of squares due to the mean, 260	
	c. Coefficient of determination, 261	
8.5.	Partitioning the model	262
	a. More than one kind of parameter, 262	
	b. Estimation, 263	
	c. Sums of squares, 264	
	d. Examples, 265	
	-i. The 1-way classification with covariate, 265	
	-ii. The 1-way classification, 265	



## 8. A General Linear Model (Continued)

- e. *Expected sums of squares*, 267
  - f. *The "invert part of the inverse" algorithm*, 267
    - i. *For the full rank case*, 267
    - ii. *For the non full rank case—a caution*, 269
  - g. *Extended partitioning*, 272
  - h. *Partitioning the total sum of squares*, 272
  - i. *Summary*, 273
- 8.6. *F-statistics from partitioned models* 274
  - a. *Error sums of squares*, 275
  - b. *Reductions in sums of squares*, 275
    - i. *Two sums of squares*, 275
    - ii.  $\chi^2$ -*properties*, 276
    - iii. *Independence of SSE*, 276
    - iv. *Independence of each other*, 277
    - v. *Summary*, 278
  - c. *A general hypothesis*, 278
  - d. *Specific hypotheses*, 279
  - e. *The full rank case*, 281
  - f. *Examples*, 281
    - i. *The 1-way classification*, 281
    - ii. *The 1-way classification with a covariate*, 281
- 8.7. *Estimable functions* 282
  - a. *Invariance to solutions of the normal equations*, 282
  - b. *Definitions*, 284
  - c. *BLU estimation*, 285
  - d. *Confidence intervals*, 286
  - e. *Other properties*, 286
  - f. *Basic estimable functions*, 287
  - g. *Full rank models*, 288
  - h. *Summary*, 288
- 8.8. *The general linear hypothesis* 288
  - a. *A general form*, 289
  - b. *The F-statistic*, 290
  - c. *The hypothesis  $H: K'\beta = 0$* , 293
  - d. *Estimation under the hypothesis*, 294
  - e. *Calculating Q*, 294
  - f. *Analysis of variance*, 296
  - g. *Non-testable hypotheses*, 296
  - h. *Partially testable hypotheses*, 298
  - i. *Independent and orthogonal contrasts*, 299
    - i. *An example (Example 4)*, 299
    - ii. *The general case*, 300
    - iii. *A necessary condition*, 301
    - iv. *Contrasts are linearly independent of the mean*, 303

8.9.	Restricted models	303
a.	<i>Using restrictions explicitly</i> , 304	
b.	<i>General methodology</i> , 305	
c.	<i>Non-estimable restrictions</i> , 306	
-i.	<i>A solution vector</i> , 306	
-ii.	<i>Estimable functions</i> , 307	
-iii.	<i>Hypothesis testing</i> , 308	
d.	<i>Estimable restrictions</i> , 309	
-i.	<i>A solution vector</i> , 309	
-ii.	<i>Estimable functions</i> , 310	
-iii.	<i>Hypothesis testing</i> , 310	
e.	<i>The full rank model</i> , 311	
8.10.	Application to a cell means model	312
8.11.	Four methods of estimation	315
8.12.	Summary	317
8.13.	Appendix	320
a.	<i>Estimation under the hypothesis (Section 8.8d)</i> , 320	
b.	<i>The likelihood ratio test</i> , 320	
8.14.	Exercises	322
<b>9.</b>	<b>The 2-Way Crossed Classification: Overparameterized Models</b>	<b>326</b>
9.1.	The with-interaction model	326
a.	<i>The model</i> , 326	
b.	<i>Relationships to cell means</i> , 327	
c.	<i>Reparameterization</i> , 328	
d.	<i>Estimable functions</i> , 331	
e.	<i>Estimation</i> , 332	
f.	<i>Hypothesis testing</i> , 332	
g.	<i>Analysis of variance</i> , 333	
h.	<i>Model equations</i> , 335	
i.	<i>The normal equations</i> , 338	
j.	<i>Solving the normal equations</i> , 339	
k.	<i>A zero sum of squares</i> , 339	
9.2.	The without-interaction model	340
a.	<i>The model</i> , 340	
b.	<i>Relationships to cell means</i> , 341	
c.	<i>Reparameterization</i> , 341	
d.	<i>Estimable functions</i> , 341	
e.	<i>Estimation</i> , 342	
f.	<i>Hypothesis testing</i> , 342	
g.	<i>Analysis of variance</i> , 343	
h.	<i>Model equations</i> , 344	
i.	<i>The normal equations</i> , 344	

<b>9. The 2-Way Overparameterized Model (Continued)</b>	
<i>j. Solving the normal equations, 344</i>	
<i>k. Sampling variances, 348</i>	
<i>l. Estimating cell means and contrasts, 349</i>	
9.3. Sums of squares for the overparameterized model	352
<i>a. <math>R(\mu)</math>, <math>R(\alpha \mu)</math> and <math>R(\beta \mu)</math>, 352</i>	
<i>b. <math>R(\beta \mu, \alpha)</math> and <math>R(\alpha \mu, \beta)</math>, 352</i>	
<i>c. <math>R(\phi \mu, \alpha, \beta)</math>, 354</i>	
<i>d. Example (continued), 355</i>	
<i>e. Analyses of variance, 357</i>	
9.4. Models with $\Sigma$ -restrictions	358
<i>a. The no-interaction model, 359</i>	
<i>-i. Solutions, 360</i>	
<i>-ii. Analysis of variance sums of squares, 360</i>	
<i>-iii. Another sum of squares, 361</i>	
<i>-iv. Associated hypotheses, 362</i>	
<i>-v. The invert-part-of-the-inverse algorithm, 362</i>	
<i>b. The with-interaction model, 363</i>	
<i>-i. All-cells-filled data, 363</i>	
<i>-ii. Some-cells-empty data, 367</i>	
<i>c. Other restrictions, 373</i>	
9.5. Constraints on solutions	373
<i>a. Solving the normal equations, 373</i>	
<i>b. A general algorithm, 375</i>	
<i>c. Other procedures, 375</i>	
<i>d. Constraints and restrictions, 376</i>	
9.6. Exercises	378
<b>10. Extended Cell Means Models</b>	<b>384</b>
10.1. Multi-factor data: basic results	384
<i>a. The model, 384</i>	
<i>b. Estimation, 385</i>	
<i>c. Residual variance, 385</i>	
<i>d. Estimable functions, 385</i>	
<i>e. Hypothesis testing, 386</i>	
<i>f. Analogy: the 1-way classification, 386</i>	
<i>g. Some-cells-empty data, 386</i>	
10.2. Multi-factor data: all-cells-filled data	387
<i>a. The 3-way crossed classification, 388</i>	
<i>b. Computing, for <math>\Sigma</math>-restricted, overparameterized models, 389</i>	
<i>c. Example, 391</i>	
<i>d. More than three factors, 395</i>	

10.3.	Main-effects-only models	396
10.4.	Models with not all interactions	400
	<ul style="list-style-type: none"> <li>a. <i>The 2-way classification</i>, 400</li> <li>b. <i>Unrestricted models</i>, 401</li> <li>c. <i>Restricted models</i>, 402 <ul style="list-style-type: none"> <li>-i. <i>Examples</i>, 402</li> <li>-ii. <i>The general case</i>, 405</li> </ul> </li> <li>d. <i>Estimability</i>, 407 <ul style="list-style-type: none"> <li>-i. <i>All cells filled</i>, 407</li> <li>-ii. <i>Some cells empty</i>, 407</li> </ul> </li> <li>e. <i>Hypothesis testing</i>, 408</li> <li>f. <i>Examples of estimable functions</i>, 409 <ul style="list-style-type: none"> <li>-i. <i>The <math>2 \times 2</math> case with all cells filled</i>, 409</li> <li>-ii. <i>The <math>2 \times 2</math> case with an empty cell</i>, 410</li> <li>-iii. <i>A 3-factor case, with no 3-factor interactions</i>, 410</li> <li>-iv. <i>A 3-factor case with two sets of interactions absent</i>, 414</li> <li>-v. <i>Conclusions</i>, 414</li> </ul> </li> </ul>	
10.5.	Exercises	415

<b>11.</b>	<b>Models with Covariables: The General Case and Some Applications</b>	<b>416</b>
11.1.	A traditional description	416
11.2.	A linear model description	419
	<ul style="list-style-type: none"> <li>a. <i>A general model</i>, 420</li> <li>b. <i>Estimation</i>, 421 <ul style="list-style-type: none"> <li>-i. <i>Normal equations</i>, 421</li> <li>-ii. <i>Assumptions on the covariates</i>, 421</li> <li>-iii. <i>Estimators</i>, 422</li> <li>-iv. <i>Dispersion matrices of estimators</i>, 423</li> </ul> </li> <li>c. <i>Analysis of variance</i>, 424</li> <li>d. <i>Hypothesis testing</i>, 424 <ul style="list-style-type: none"> <li>-i. <i>Some general hypotheses</i>, 424</li> <li>-ii. <i>Associated hypotheses in the analysis of variance</i>, 426</li> </ul> </li> <li>e. <i>Examples: 1-way classification, one covariable</i>, 426 <ul style="list-style-type: none"> <li>-i. <i>The single slope model</i>, 426</li> <li>-ii. <i>The multiple slopes model</i>, 429</li> <li>-iii. <i>Other hypotheses about slopes</i>, 430</li> </ul> </li> </ul>	
11.3.	Confirming associated hypotheses of Chapter 6	431
	<ul style="list-style-type: none"> <li>a. <i>The single slope model (Table 6.4)</i>, 432 <ul style="list-style-type: none"> <li>-i. <i>Sums of squares</i>, 432</li> <li>-ii. <i>Associated hypotheses</i>, 432</li> </ul> </li> </ul>	

## 11. Covariables (Continued)

- b. The multiple slopes model: Tables 6.6 and 6.7, 434*
      - i. Models, 434*
      - ii. Sums of squares, 436*
      - iii. Associated hypotheses, 436*
    - c. Usefulness of the hypotheses, 438*
  - 11.4. The 1-way classification: restricted overparameterized models 438
  - 11.5. The 1-way classification: two covariates 441
    - a. Basic tools and notation, 441*
    - b. Single slope for each covariate, 442*
    - c. Intra-class slopes and a single slope, 444*
    - d. Intra-class slopes for each covariate, 445*
  - 11.6. The 2-way classification: single slope models for one covariate 447
    - a. The with-interaction model, 447*
    - b. The no-interaction model, 448*
  - 11.7. The 2-way classification: multiple slope models 451
    - a. Interaction models, with unbalanced data, 451*
      - i. Intra-row slopes for one covariate, 451*
      - ii. Intra-row slopes for one covariate, intra-column for another, 452*
      - iii. Intra-row slopes for each of two covariates, 452*
      - iv. Intra-row plus intra-column slopes for one covariate, 452*
    - b. No-interaction models with balanced data, 452*
      - i. Intra-row slopes for one covariate, 453*
      - ii. Intra-row slopes for one covariate, intra-column for another, 453*
      - iii. Intra-row slopes for each of two covariates, 454*
      - iv. Intra-row plus intra-column slopes for one covariate, 454*
  - 11.8. Exercises 454

## 12. Comments on Computing Packages 457

- 12.1. Sums of squares output 458
  - a.  $\Sigma$ -restrictions, 460*
  - b. Weighted squares of means, 461*
  - c. Patterns of filled cells, 461*
  - d. Balanced data, 461*
  - e. Four sets of output from SAS GLM, 461*
- 12.2. Sums of squares from SAS GLM 461
  - a. Type I, 462*
  - b. Type II, 462*
  - c. Type III, 463*
  - d. Type IV, 463*

12.3.	Estimable functions in SAS GLM output	465
	<ul style="list-style-type: none"> <li>a. <i>Examples</i>, 465</li> <li>b. <i>Output is parameter labels and coefficients</i>, 465</li> <li>c. <i>Estimable function obtained from output</i>, 466</li> <li>d. <i>Estimable function provides the hypothesis</i>, 467</li> <li>e. <i>Summary</i>, 468</li> <li>f. <i>Comment</i>, 468</li> <li>g. <i>Verification</i>, 469 <ul style="list-style-type: none"> <li>-i. <i>A general result</i>, 469</li> <li>-ii. <i>Examples</i>, 470</li> </ul> </li> </ul>	
12.4.	Solution vector output	471
	<ul style="list-style-type: none"> <li>a. <i>BMDP2V</i>, 472</li> <li>b. <i>GENSTAT ANOVA</i>, 472</li> <li>c. <i>SAS GLM</i>, 472 <ul style="list-style-type: none"> <li>-i. <i>Using a generalized inverse</i>, 472</li> <li>-ii. <i>The solution vector</i>, 472</li> <li>-iii. <i>Standard errors and t-statistics output</i>, 473</li> <li>-iv. <i>Estimating estimable functions</i>, 474</li> <li>-v. <i>Other features</i>, 474</li> </ul> </li> <li>d. <i>SAS HARVEY</i>, 475</li> <li>e. <i>SPSS ANOVA</i>, 477</li> </ul>	
12.5.	A faulty computing algorithm	478
12.6.	Output for analysis with covariates	480
<b>13.</b>	<b>Mixed Models: A Thumbnail Survey</b>	<b>484</b>
13.1.	Introduction	484
	<ul style="list-style-type: none"> <li>a. <i>An example</i>, 484</li> <li>b. <i>A general description</i>, 486</li> <li>c. <i>Estimation</i>, 489</li> </ul>	
13.2.	Estimating variance components from balanced data	490
	<ul style="list-style-type: none"> <li>a. <i>Example</i>, 491</li> <li>b. <i>Merits</i>, 492 <ul style="list-style-type: none"> <li>-i. <i>Broad applicability</i>, 492</li> <li>-ii. <i>Unbiasedness</i>, 492</li> <li>-iii. <i>Sampling variances</i>, 493</li> </ul> </li> <li>c. <i>Demerits</i>, 494 <ul style="list-style-type: none"> <li>-i. <i>Negative estimates</i>, 494</li> <li>-ii. <i>Distributional properties</i>, 495</li> </ul> </li> </ul>	
13.3.	Estimating variance components from unbalanced data	495
	<ul style="list-style-type: none"> <li>a. <i>General ANOVA methodology</i>, 495 <ul style="list-style-type: none"> <li>-i. <i>Merits</i>, 498</li> <li>-ii. <i>Demerits</i>, 498</li> </ul> </li> </ul>	

<b>13. Mixed Models (Continued)</b>	
<i>b. Henderson's Method I, 498</i>	
<i>c. Henderson's Method II, 500</i>	
<i>d. Henderson's Method III, 500</i>	
<i>e. ML (Maximum Likelihood), 502</i>	
<i>f. REML (Restricted Maximum Likelihood), 504</i>	
<i>g. MINQUE (Minimum Norm Quadratic Unbiased Estimation), 506</i>	
<i>h. MIVQUE (Minimum Variance Quadratic Unbiased Estimation), 507</i>	
<i>i. I-MINQUE (Iterative MINQUE), 507</i>	
13.4. Prediction of random variables	507
13.5. An example: the 1-way classification	510
<i>a. The model, 510</i>	
<i>b. Analysis of variance, 511</i>	
<i>c. Testing the mean, 511</i>	
<i>d. Prediction, 513</i>	
<i>e. Estimating variance components, 514</i>	
13.6. Exercises	514
<b>References</b>	<b>517</b>
<b>Statistical Tables</b>	<b>522</b>
<b>List of Tables and Figures</b>	<b>527</b>
<b>Index</b>	<b>533</b>