

A Practical Course in Differential Equations and Mathematical Modelling

Classical and New Methods
Nonlinear Mathematical Models
Symmetry and Invariance Principles

Nail H. Ibragimov



Higher Education Press



World Scientific

Contents

Preface

1	Selected topics from analysis	1
1.1	Elementary mathematics	1
1.1.1	Numbers, variables and elementary functions	1
1.1.2	Quadratic and cubic equations	5
1.1.3	Areas of similar figures. Ellipse as an example	8
1.1.4	Algebraic curves of the second degree	9
1.2	Differential and integral calculus	14
1.2.1	Rules for differentiation	14
1.2.2	The mean value theorem	15
1.2.3	Invariance of the differential	15
1.2.4	Rules for integration	16
1.2.5	The Taylor series	17
1.2.6	Complex variables	19
1.2.7	Approximate representation of functions	20
1.2.8	Jacobian. Functional independence. Change of variables in multiple integrals	21
1.2.9	Linear independence of functions. Wronskian	22
1.2.10	Integration by quadrature	23
1.2.11	Differential equations for families of curves	24
1.3	Vector analysis	26
1.3.1	Vector algebra	26
1.3.2	Vector functions	28
1.3.3	Vector fields	29
1.3.4	Three classical integral theorems	30
1.3.5	The Laplace equation	31
1.3.6	Differentiation of determinants	32
1.4	Notation of differential algebra	32
1.4.1	Differential variables. Total differentiation	32
1.4.2	Higher derivatives of the product and of composite functions	33
1.4.3	Differential functions with several variables	34

1.4.4	The frame of differential equations	35
1.4.5	Transformation of derivatives	36
1.5	Variational calculus	38
1.5.1	Principle of least action	38
1.5.2	Euler-Lagrange equations with several variables	39
	Problems to Chapter 1	40
2	Mathematical models	45
2.1	Introduction	45
2.2	Natural phenomena	46
2.2.1	Population models	46
2.2.2	Ecology: Radioactive waste products	47
2.2.3	Kepler's laws. Newton's gravitation law	48
2.2.4	Free fall of a body near the earth	50
2.2.5	Meteoroid	50
2.2.6	A model of rainfall	52
2.3	Physics and engineering sciences	53
2.3.1	<i>Newton's model of cooling</i>	53
2.3.2	Mechanical vibrations. Pendulum	60
2.3.3	Collapse of driving shafts	64
2.3.4	The van der Pol equation	66
2.3.5	Telegraph equation	67
2.3.6	Electrodynamics	68
2.3.7	The Dirac equation	69
2.3.8	Fluid dynamics	69
2.3.9	The Navier-Stokes equations	71
2.3.10	A model of an irrigation system	71
2.3.11	Magnetohydrodynamics	71
2.4	Diffusion phenomena	72
2.4.1	Linear heat equation	72
2.4.2	Nonlinear heat equation	74
2.4.3	The Burgers and Korteweg-de Vries equations	75
2.4.4	Mathematical modelling in finance	75
2.5	Biomathematics	76
2.5.1	Smart mushrooms	76
2.5.2	A tumour growth model	78
2.6	Wave phenomena	79
2.6.1	Small vibrations of a string	79
2.6.2	Vibrating membrane	82
2.6.3	Minimal surfaces	84
2.6.4	Vibrating slender rods and plates	84
2.6.5	Nonlinear waves	86
2.6.6	<i>The Chaplygin and Tricomi equations</i>	88
	Problems to Chapter 2	88

3	Ordinary differential equations: Traditional approach	91
3.1	Introduction and elementary methods	91
3.1.1	Differential equations. Initial value problem	91
3.1.2	Integration of the equation $y^{(n)} = f(x)$	93
3.1.3	Homogeneous equations	93
3.1.4	Different types of homogeneity	96
3.1.5	Reduction of order	97
3.1.6	Linearization through differentiation	98
3.2	First-order equations	99
3.2.1	Separable equations	99
3.2.2	Exact equations	99
3.2.3	Integrating factor (A. Clairaut, 1739)	101
3.2.4	The Riccati equation	102
3.2.5	The Bernoulli equation	106
3.2.6	Homogeneous linear equations	106
3.2.7	Non-homogeneous linear equations. Variation of the parameter	107
3.3	Second-order linear equations	108
3.3.1	Homogeneous equation: Superposition	109
3.3.2	Homogeneous equation: Equivalence properties	110
3.3.3	Homogeneous equation: Constant coefficients	113
3.3.4	Non-homogeneous equation: Variation of parameters	115
3.3.5	Bessel's equation and the Bessel functions	119
3.3.6	Hypergeometric equation	119
3.4	Higher-order linear equations	120
3.4.1	Homogeneous equations. Fundamental system	121
3.4.2	Non-homogeneous equations. Variation of parameters	121
3.4.3	Equations with constant coefficients	122
3.4.4	Euler's equation	123
3.5	Systems of first-order equations	124
3.5.1	General properties of systems	124
3.5.2	First integrals	125
3.5.3	Linear systems with constant coefficients	129
3.5.4	Variation of parameters for systems	130
	Problems to Chapter 3	133
4	First-order partial differential equations	135
4.1	Introduction	135
4.2	Homogeneous linear equation	136
4.3	Particular solutions of non-homogeneous equations	138
4.4	Quasi-linear equations	139
4.5	Systems of homogeneous equations	142
	Problems to Chapter 4	145

5	Linear partial differential equations of the second order	149
5.1	Equations with several variables	149
5.1.1	Classification at a fixed point	149
5.1.2	Adjoint linear differential operators	151
5.2	Classification of equations in two independent variables	153
5.2.1	Characteristics. Three types of equations	153
5.2.2	The standard form of the hyperbolic equations	155
5.2.3	The standard form of the parabolic equations	156
5.2.4	The standard form of the elliptic equations	157
5.2.5	Equations of a mixed type	158
5.2.6	The type of nonlinear equations	158
5.3	Integration of hyperbolic equations in two variables	159
5.3.1	d'Alembert's solution	159
5.3.2	Equations reducible to the wave equation	160
5.3.3	Euler's method	165
5.3.4	Laplace's cascade method	167
5.4	The initial value problem	169
5.4.1	The wave equation	169
5.4.2	Non-homogeneous wave equation	170
5.5	Mixed problem. Separation of variables	172
5.5.1	Vibration of a string tied at its ends	172
5.5.2	Mixed problem for the heat equation	175
	Problems to Chapter 5	177
6	Nonlinear ordinary differential equations	179
6.1	Introduction	179
6.2	Transformation groups	180
6.2.1	One-parameter groups on the plane	180
6.2.2	Group generator and the Lie equations	181
6.2.3	Exponential map	183
6.2.4	Invariants and invariant equations	184
6.2.5	Canonical variables	187
6.3	Symmetries of first-order equations	188
6.3.1	First prolongation of group generators	188
6.3.2	Symmetry group: definition and main property	189
6.3.3	Equations with a given symmetry	191
6.4	Integration of first-order equations using symmetries	193
6.4.1	Lie's integrating factor	193
6.4.2	Integration using canonical variables	196
6.4.3	Invariant solutions	199
6.4.4	General solution provided by invariant solutions	200
6.5	Second-order equations	201
6.5.1	Second prolongation of group generators. Calculation of symmetries	201

CONTENTS

6.5.2	Lie algebras	204
6.5.3	Standard forms of two-dimensional Lie algebras	205
6.5.4	Lie's integration method	206
6.5.5	Integration of linear equations with a known particular solution	212
6.5.6	Lie's linearization test	214
6.6	Higher-order equations	218
6.6.1	Invariant solutions. Derivation of Euler's ansatz	218
6.6.2	Integrating factor (N.H. Ibragimov, 2006)	220
6.6.3	Linearization of third-order equations	228
6.7	Nonlinear superposition	235
6.7.1	Introduction	235
6.7.2	Main theorem on nonlinear superposition	236
6.7.3	Examples of nonlinear superposition	241
6.7.4	Integration of systems using nonlinear superposition	249
	Problems to Chapter 6	251
7	Nonlinear partial differential equations	255
7.1	Symmetries	255
7.1.1	Definition and calculation of symmetry groups	256
7.1.2	Group transformations of solutions	260
7.2	Group invariant solutions	262
7.2.1	Introduction	262
7.2.2	The Burgers equation	263
7.2.3	A nonlinear boundary-value problem	266
7.2.4	Invariant solutions for an irrigation system	268
7.2.5	Invariant solutions for a tumour growth model	270
7.2.6	An example from nonlinear optics	272
7.3	Invariance and conservation laws	274
7.3.1	Introduction	274
7.3.2	Preliminaries	277
7.3.3	Noether's theorem	278
7.3.4	Higher-order Lagrangians	278
7.3.5	Conservation theorems for ODEs	279
7.3.6	Generalization of Noether's theorem	280
7.3.7	Examples from classical mechanics	281
7.3.8	Derivation of Einstein's formula for energy	284
7.3.9	Conservation laws for the Dirac equations	285
	Problems to Chapter 7	286
8	Generalized functions or distributions	291
8.1	Introduction of generalized functions	291
8.1.1	Heuristic considerations	292
8.1.2	Definition and examples of distributions	293
8.1.3	Representations of the δ -function as a limit	294

8.2	Operations with distributions	295
8.2.1	Multiplication by a function	295
8.2.2	Differentiation	296
8.2.3	Direct product of distributions	296
8.2.4	Convolution	297
8.3	The distribution $\Delta(r^{2-n})$	298
8.3.1	The mean value over the sphere	298
8.3.2	Solution of the Laplace equation $\Delta v(r) = 0$	298
8.3.3	Evaluation of the distribution $\Delta(r^{2-n})$	299
8.4	Transformations of distributions	301
8.4.1	Motivation by linear transformations	301
8.4.2	Change of variables in the δ -function	302
8.4.3	Arbitrary group transformations	302
8.4.4	Infinitesimal transformation of distributions	304
	Problems to Chapter 8	304
9	Invariance principle and fundamental solutions	307
9.1	Introduction	307
9.2	The invariance principle	308
9.2.1	Formulation of the invariance principle	308
9.2.2	Fundamental solution of linear equations with constant coefficients	308
9.2.3	Application to the Laplace equation	309
9.2.4	Application to the heat equation	311
9.3	Cauchy's problem for the heat equation	313
9.3.1	Fundamental solution for the Cauchy problem	313
9.3.2	Derivation of the fundamental solution for the Cauchy problem from the invariance principle	313
9.3.3	Solution of the Cauchy problem	315
9.4	Wave equation	316
9.4.1	Preliminaries on differential forms	316
9.4.2	Auxiliary equations with distributions	320
9.4.3	Symmetries and definition of fundamental solutions for the wave equation	321
9.4.4	Derivation of the fundamental solution	323
9.4.5	Solution of the Cauchy problem	324
9.5	Equations with variable coefficients	325
	Problems to Chapter 9	325
	Answers	327
	Bibliography	337
	Index	341