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GEOMETRIC ALGEBRA

The Morgan Kaufmann Series in Computer Graphics
for Computer Science

AN OBJECT-ORIENTED APPROACH TO GEOMETRY



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