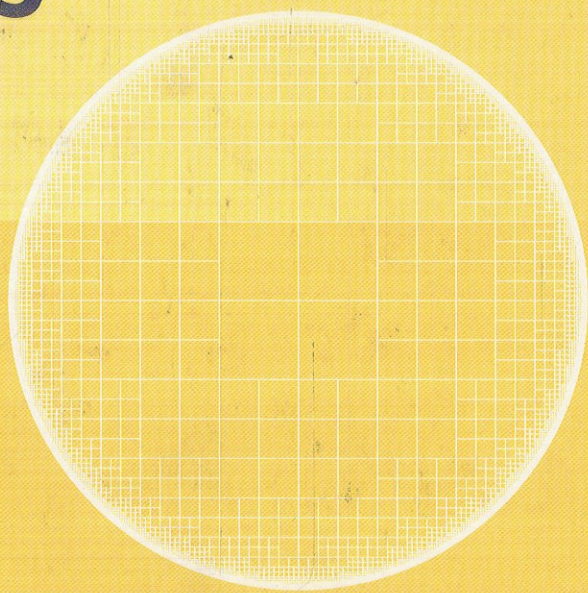


# Graduate Texts in Mathematics

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## Modern Fourier Analysis

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