

A Guided Tour of
**Mathematical Methods
for the Physical Sciences**

Second Edition



Roel Snieder

CAMBRIDGE

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