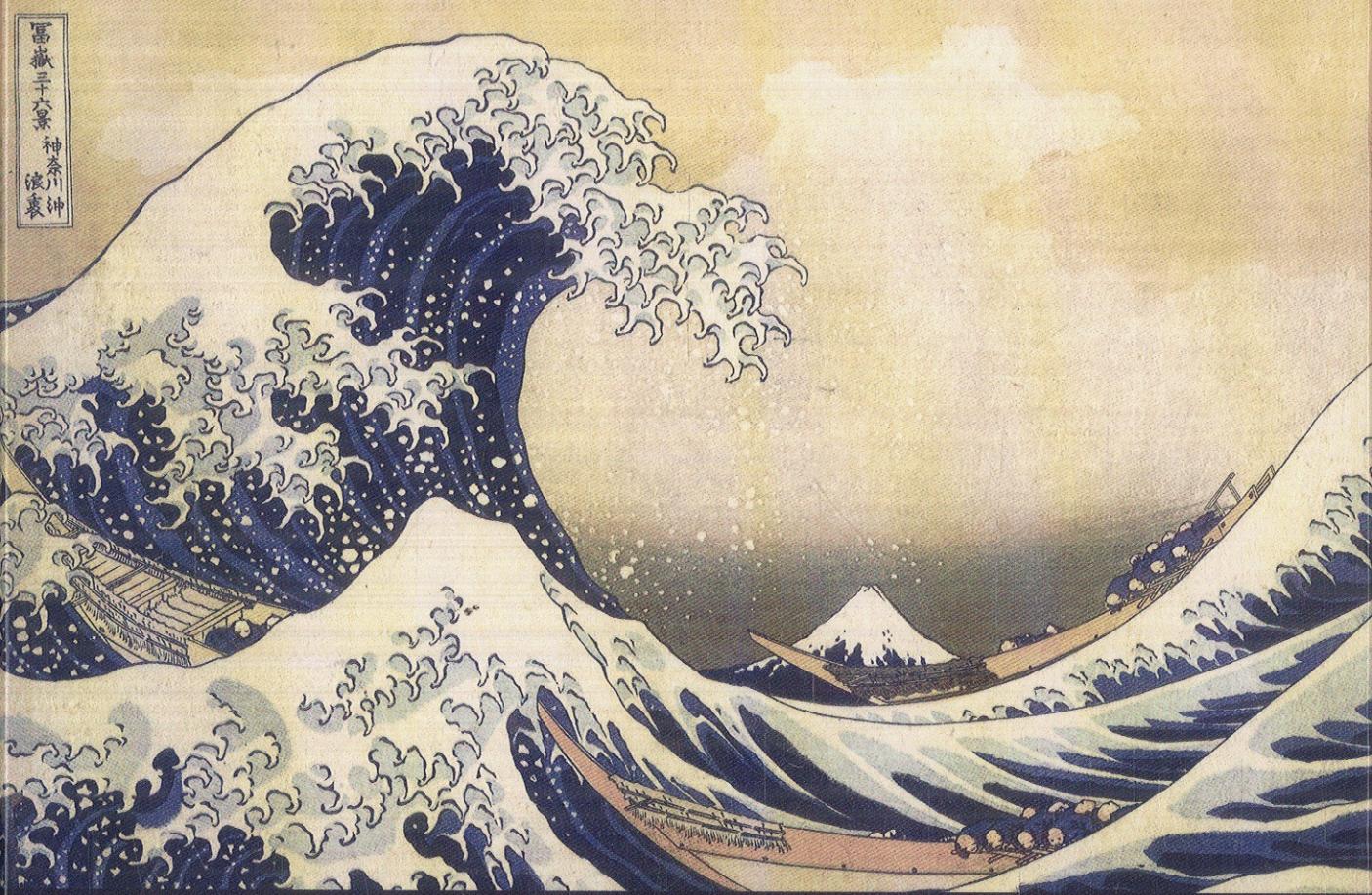


a wavelet tour of signal processing

The Sparse Way



Third Edition

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