

DISCRETE MATHEMATICS AND ITS APPLICATIONS

Series Editor KENNETH H. ROSEN

# HANDBOOK OF MATHEMATICAL INDUCTION

## THEORY AND APPLICATIONS



David S. Gunderson



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