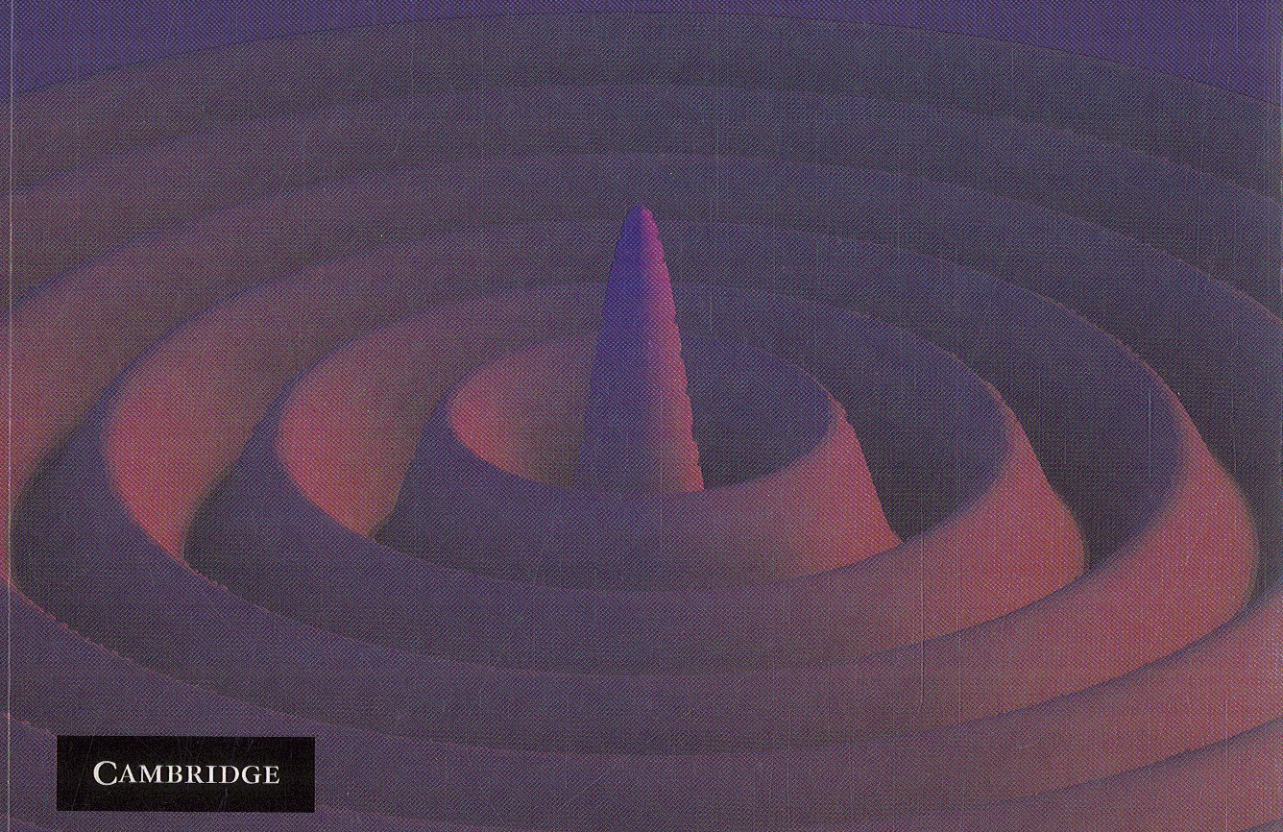


Linear Partial Differential Equations and Fourier Theory

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An abstract graphic at the bottom of the cover. It features a series of concentric, slightly raised rings that spiral inward toward a central point. In the center, there is a small, sharp, cone-like structure. The entire graphic is rendered in shades of purple, blue, and red, with a textured, almost woven appearance.

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