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# Geometric Computing

## For Wavelet Transforms, Robot Vision, Learning, Control and Action

Springer

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