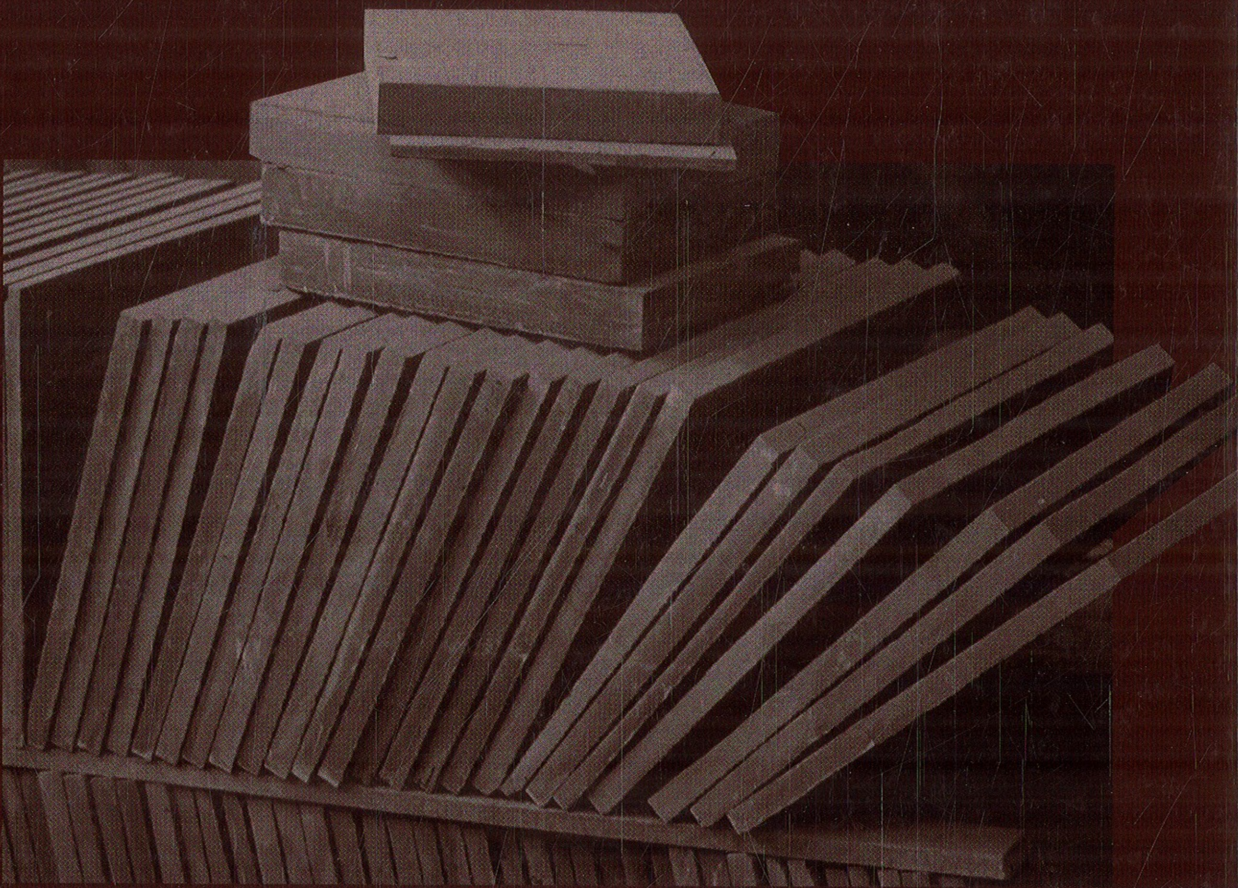
Computational Complexity

A Modern Approach



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Contents

About this book — *page* xiii
Acknowledgments — xvii
Introduction — xix

0 Notational conventions 1
- 0.1 Representing objects as strings — 2
- 0.2 Decision problems/languages — 3
- *0.3 Big-oh notation* — *3*
- EXERCISES — 4

PART ONE: BASIC COMPLEXITY CLASSES — 7

1 The computational model—and why it doesn't matter 9
- 1.1 Modeling computation: What you really need to know — 10
- 1.2 The Turing machine — 11
- 1.3 Efficiency and running time — 15
- 1.4 Machines as strings and the universal Turing machine — 19
- 1.5 Uncomputability: An introduction — 21
- 1.6 The Class P — 24
- 1.7 Proof of Theorem 1.9: Universal simulation in $O(T \log T)$-time — 29
- CHAPTER NOTES AND HISTORY — 32
- EXERCISES — 34

2 NP and NP completeness 38
- 2.1 The Class **NP** — 39
- 2.2 Reducibility and NP-completeness — 42
- 2.3 The Cook-Levin Theorem: Computation is local — 44
- 2.4 The web of reductions — 50
- 2.5 Decision versus search — 54
- 2.6 **coNP**, **EXP**, and **NEXP** — 55
- 2.7 More thoughts about **P**, **NP**, and all that — 57
- CHAPTER NOTES AND HISTORY — 62
- *EXERCISES* — 63

3 Diagonalization 68
3.1 Time Hierarchy Theorem 69
3.2 Nondeterministic Time Hierarchy Theorem 69
3.3 Ladner's Theorem: Existence of **NP**-intermediate problems 71
3.4 Oracle machines and the limits of diagonalization 72
CHAPTER NOTES AND HISTORY 76
EXERCISES 77

4 Space complexity 78
4.1 Definition of space-bounded computation 78
4.2 **PSPACE** completeness 83
4.3 **NL** completeness 87
CHAPTER NOTES AND HISTORY 93
EXERCISES 93

5 The polynomial hierarchy and alternations 95
5.1 The Class Σ_2^p 96
5.2 The polynomial hierarchy 97
5.3 Alternating Turing machines 99
5.4 Time versus alternations: Time-space tradeoffs for SAT 101
5.5 Defining the hierarchy via oracle machines 102
CHAPTER NOTES AND HISTORY 104
EXERCISES 104

6 Boolean circuits 106
6.1 Boolean circuits and $\mathbf{P}_{/\mathbf{poly}}$ 107
6.2 Uniformly generated circuits 111
6.3 Turing machines that take advice 112
6.4 $\mathbf{P}_{/\mathbf{poly}}$ and **NP** 113
6.5 Circuit lower bounds 115
6.6 Nonuniform Hierarchy Theorem 116
6.7 Finer gradations among circuit classes 116
6.8 Circuits of exponential size 119
CHAPTER NOTES AND HISTORY 120
EXERCISES 121

7 Randomized computation 123
7.1 Probabilistic Turing machines 124
7.2 Some examples of PTMs 126
7.3 One-sided and "zero-sided" error: **RP**, **coRP**, **ZPP** 131
7.4 The robustness of our definitions 132
7.5 Relationship between **BPP** and other classes 135
7.6 Randomized reductions 138
7.7 Randomized space-bounded computation 139
CHAPTER NOTES AND HISTORY 140
EXERCISES 141

8 Interactive proofs 143
8.1 Interactive proofs: Some variations 144
8.2 Public coins and **AM** 150
8.3 **IP = PSPACE** 157
8.4 The power of the prover 162
8.5 Multiprover interactive proofs (MIP) 163
8.6 Program checking 164
8.7 Interactive proof for the permanent 167
CHAPTER NOTES AND HISTORY 169
EXERCISES 170

9 Cryptography 172
9.1 Perfect secrecy and its limitations 173
9.2 Computational security, one-way functions, and pseudorandom generators 175
9.3 Pseudorandom generators from one-way permutations 180
9.4 Zero knowledge 186
9.5 Some applications 189
CHAPTER NOTES AND HISTORY 194
EXERCISES 197

10 Quantum computation 201
10.1 Quantum weirdness: The two-slit experiment 202
10.2 Quantum superposition and qubits 204
10.3 Definition of quantum computation and BQP 209
10.4 Grover's search algorithm 216
10.5 Simon's algorithm 219
10.6 Shor's algorithm: Integer factorization using quantum computers 221
10.7 BQP and classical complexity classes 230
CHAPTER NOTES AND HISTORY 232
EXERCISES 234

11 PCP theorem and hardness of approximation: An introduction 237
11.1 Motivation: Approximate solutions to **NP**-hard optimization problems 238
11.2 Two views of the **PCP** Theorem 240
11.3 Equivalence of the two views 244
11.4 Hardness of approximation for vertex cover and independent set 247
11.5 $\mathbf{NP} \subseteq \mathbf{PCP}(\text{poly}(n), 1)$: **PCP** from the Walsh-Hadamard code 249
CHAPTER NOTES AND HISTORY 254
EXERCISES 255

PART TWO: LOWER BOUNDS FOR CONCRETE COMPUTATIONAL MODELS 257

12 Decision trees 259
12.1 Decision trees and decision tree complexity 259
12.2 Certificate complexity 262
12.3 Randomized decision trees 263

12.4 Some techniques for proving decision tree lower bounds 264
CHAPTER NOTES AND HISTORY 268
EXERCISES 269

13 Communication complexity 270
13.1 Definition of two-party communication complexity 271
13.2 Lower bound methods 272
13.3 Multiparty communication complexity 278
13.4 Overview of other communication models 280
CHAPTER NOTES AND HISTORY 282
EXERCISES 283

14 Circuit lower bounds: Complexity theory's Waterloo 286
14.1 AC^0 and Håstad's Switching Lemma 286
14.2 Circuits with "counters": ACC 291
14.3 Lower bounds for monotone circuits 293
14.4 Circuit complexity: The frontier 297
14.5 Approaches using communication complexity 300
CHAPTER NOTES AND HISTORY 304
EXERCISES 305

15 Proof complexity 307
15.1 Some examples 307
15.2 Propositional calculus and resolution 309
15.3 Other proof systems: A tour d'horizon 313
15.4 Metamathematical musings 315
CHAPTER NOTES AND HISTORY 316
EXERCISES 317

16 Algebraic computation models 318
16.1 Algebraic straight-line programs and algebraic circuits 319
16.2 Algebraic computation trees 326
16.3 The Blum-Shub-Smale model 331
CHAPTER NOTES AND HISTORY 334
EXERCISES 336

PART THREE: ADVANCED TOPICS 339

17 Complexity of counting 341
17.1 Examples of counting problems 342
17.2 The Class #P 344
17.3 #P completeness 345
17.4 Toda's theorem: $\mathbf{PH} \subseteq \mathbf{P}^{\#\mathsf{SAT}}$ 352
17.5 Open problems 358
CHAPTER NOTES AND HISTORY 359
EXERCISES 359

18 Average case complexity: Levin's theory 361
18.1 Distributional problems and distP 362
18.2 Formalization of "real-life distributions" 365
18.3 dist**np** and its complete problems 365
18.4 Philosophical and practical implications 369
CHAPTER NOTES AND HISTORY 371
EXERCISES 371

19 Hardness amplification and error-correcting codes 373
19.1 Mild to strong hardness: Yao's XOR lemma 375
19.2 Tool: Error-correcting codes 379
19.3 Efficient decoding 385
19.4 Local decoding and hardness amplification 386
19.5 List decoding 392
19.6 Local list decoding: Getting to **BPP** = **P** 394
CHAPTER NOTES AND HISTORY 398
EXERCISES 399

20 Derandomization 402
20.1 Pseudorandom generators and derandomization 403
20.2 Proof of Theorem 20.6: Nisan-Wigderson Construction 407
20.3 Derandomization under uniform assumptions 413
20.4 Derandomization requires circuit lower bounds 415
CHAPTER NOTES AND HISTORY 418
EXERCISES 419

21 Pseudorandom constructions: Expanders and extractors 421
21.1 Random walks and eigenvalues 422
21.2 Expander graphs 426
21.3 Explicit construction of expander graphs 434
21.4 Deterministic logspace algorithm for undirected connectivity 440
21.5 Weak random sources and extractors 442
21.6 Pseudorandom generators for space-bounded computation 449
CHAPTER NOTES AND HISTORY 454
EXERCISES 456

22 Proofs of PCP theorems and the Fourier transform technique 460
22.1 Constraint satisfaction problems with nonbinary alphabet 461
22.2 Proof of the PCP theorem 461
22.3 Hardness of $2CSP_W$: Tradeoff between gap and alphabet size 472
22.4 Håstad's 3-bit PCP Theorem and hardness of MAX-3SAT 474
22.5 Tool: The Fourier transform technique 475
22.6 Coordinate functions, long Code, and its testing 480
22.7 Proof of Theorem 22.16 481
22.8 Hardness of approximating SET-COVER 486
22.9 Other **PCP** theorems: A survey 488
22.A Transforming qCSP instances into "nice" instances 491
CHAPTER NOTES AND HISTORY 493
EXERCISES 495

23 Why are circuit lower bounds so difficult? 498
23.1 Definition of natural proofs 499
23.2 What's so natural about natural proofs? 500
23.3 Proof of Theorem 23.1 503
23.4 An "unnatural" lower bound 504
23.5 A philosophical view 505
CHAPTER NOTES AND HISTORY 506
EXERCISES 507

Appendix: Mathematical background 508
A.1 Sets, functions, pairs, strings, graphs, logic 509
A.2 Probability theory 510
A.3 Number theory and groups 517
A.4 Finite fields 521
A.5 Basic facts from linear Algebra 522
A.6 Polynomials 527

Hints and selected exercises 531
Main theorems and definitions 545
Bibliography 549
Index 575
Complexity class index 579