

DYNAMICAL Symmetry



Carl E Wulfman

Contents

Dedication	v
Preface	vii
Acknowledgements	xi
1. Introduction	1
1.1 On Geometric Symmetry and Invariance in the Sciences	1
1.2 Fock's Discovery	6
1.3 Keplerian Symmetry	8
1.4 Dynamical Symmetry	8
1.5 Dynamical Symmetries Responsible for Degeneracies and Their Physical Consequences	10
1.6 Dynamical Symmetries When Energies Can Vary	12
1.7 The Need for Critical Reexamination of Concepts of Physical Symmetry. Lie's Discoveries	13
Appendix A: Historical Note	18
References	19
2. Physical Symmetry and Geometrical Symmetry	23
2.1 Geometrical Interpretation of the Invariance Group of an Equation; Symmetry Groups	23
2.2 On Geometric Interpretations of Equations	29
2.3 Geometric Interpretations of Some Transformations in the Euclidean Plane	30
2.4 The Group of Linear Transformations of Two Variables	36

2.5	Physical Interpretation of Rotations	37
2.6	Intrinsic Symmetry of an Equation	39
2.7	Non-Euclidean Geometries	40
2.8	Invariance Group of a Geometry	48
2.9	Symmetry in Euclidean Spaces	49
2.10	Symmetry in the Spacetime of Special Relativity	51
2.11	Geometrical Interpretations of Nonlinear Transformations: Stereographic Projections	56
2.12	Continuous Groups that Leave Euclidean and Pseudo-Euclidean Metrics Invariant	62
2.13	Geometry, Symmetry, and Invariance	65
	Appendix A: Stereographic Projection of Circles	66
	References	68
3.	On Symmetries Associated With Hamiltonian Dynamics	71
3.1	Invariance of a Differential Equation	71
3.2	Hamilton's Equations	73
3.3	Transformations that Convert Hamilton's Equations into Hamilton's Equations; Symplectic Groups	76
3.4	Invariance Transformations of Hamilton's Equations of Motion	78
3.5	Geometrization of Hamiltonian Mechanics	80
3.6	Symmetry in Two-dimensional Symplectic Space	91
3.7	Symmetry in Two-dimensional Hamiltonian Phase Space	92
3.8	Symmetries Defined by Linear Symplectic Transformations	95
3.9	Nonlinear Transformations in Two-dimensional Phase Space	97
3.10	Dynamical Symmetries as Intrinsic Symmetries of Differential Equations and as Geometric Symmetries . .	102
	Exercises	104
	References	104
4.	One-Parameter Transformation Groups	105
4.1	Introduction	105
4.2	Finite Transformations of a Continuous Group Define Infinitesimal Transformations and Vector Fields	108
4.3	Spaces in which Transformations will be Assumed to Act	112

4.4	The Defining Equations of One-Parameter Groups of Infinitesimal Transformations. Group Generators	114
4.5	The Differential Equations that Define Infinitesimal Transformations Define Finite Transformation Groups	116
4.6	The Operator of Finite Transformations	119
4.7	Changing Variables in Group Generators	122
4.8	The Rectification Theorem	124
4.9	Conversion of Non-autonomous ODEs to Autonomous ODEs	129
4.10	N-th Order ODEs as Sets of First-order ODEs	131
4.11	Conclusion	131
	Appendix: Homeomorphisms, Diffeomorphisms, and Topology	132
	Exercises	133
	References	134

5. Everywhere-Local Invariance 135

5.1	Invariance under the Action of One-Parameter Lie Transformation Groups	135
5.2	Transformation of Infinitesimal Displacements	141
5.3	Transformations and Invariance of Work, Pfaffians, and Metrics	143
5.4	Point Transformations of Derivatives	147
5.5	Contact Transformations	149
5.6	Invariance of an Ordinary Differential Equation of First-order under Point Transformations; Extended Generators	151
5.7	Invariance of Second-order Ordinary Differential Equations under Point Transformations. Harmonic Oscillators	154
5.8	The Commutator of Two Operators	156
5.9	Invariance of Sets of ODEs. Constants of Motion	158
5.10	Conclusion	161
	Appendix A. Relation between Symmetries and Integrating Factors	162

Appendix B. Proof That The Commutator of Lie Generators is Invariant Under Diffeomorphisms	163
Appendix C. Isolating and Non-isolating Integrals of Motion	165
Exercises	167
References	167
6. Lie Transformation Groups and Lie Algebras	169
6.1 Relation of Many-Parameter Lie Transformation Groups to Lie Algebras	169
6.2 The Differential Equations that Define Many-Parameter Groups	175
6.3 Real Lie Algebras	177
6.4 Relations between Commutation Relations and the Action of Transformation Groups: Some Examples	179
6.5 Transitivity	186
6.6 Complex Lie Algebras	187
6.7 The Cartan- Killing Form; Labeling and Shift Operators	188
6.8 Casimir Operators	191
6.9 Groups That Vary the Parameters of Transformation Groups	192
6.10 Lie Symmetries Induced from Observations	193
6.11 Conclusion	195
Appendix A. Definition of Lie Groups by Partial Differential Equations	196
Appendix B. Classification of Lie Algebras and Lie Groups	202
Exercises	204
References	205
7. Dynamical Symmetry in Hamiltonian Mechanics	207
7.1 General Invariance Properties of Newtonian Mechanics	207
7.2 Relationship of Phase Space to Abstract Symplectic Space	210
7.3 Hamilton's Equations in PQ Space. Constants of Motion	214

7.4	Poisson Bracket Operators	217
7.5	Hamiltonian Dynamical Symmetries in PQ Space . . .	220
7.6	Hamilton's Equations in Classical PQET Space; Conservation Laws Arising From Galilei Invariance . . .	230
7.7	Time-dependent Constants of Motion; Dynamical Groups That Act Transitively	234
7.8	The Symplectic Groups $Sp(2n, r)$	237
7.9	Generalizations of Symplectic Groups That Have an Infinite Number of One-Parameter Groups	239
	Appendix A. Lagrange's Equation's and the Definition of Phase Space	240
	Appendix B. The Variable Conjugate to Time in PQET Space	243
	Exercises	246
	References	246
8.	Symmetries of Classical Keplerian Motion	247
8.1	Newtonian Mechanics of Planetary Motion	247
8.2	Hamiltonian Formulation of Keplerian Motions in Phase Space	253
8.3	Symmetry Coordinates For Keplerian Motions	257
8.4	Geometrical Symmetries of Bound Keplerian Systems in Phase Space	262
8.5	Symmetries of Keplerian Systems with Non-negative Energies	267
8.6	The $SO(4, 1)$ Dynamical Symmetry	270
8.7	Concluding Remarks	271
	Exercises	272
	References	273
9.	Dynamical Symmetry in Schrödinger Quantum Mechanics	275
9.1	Superposition Invariance	275
9.2	The Correspondence Principle	276
9.3	Correspondence Between Quantum Mechanical Operators and Functions of Classical Dynamical Variables	280
9.4	Lie Algebraic Extension of the Correspondence Principle	281

9.5	Some Properties of Invariance Transformations of Partial Differential Equations Relevant to Quantum Mechanics	285
9.6	Determination of Generators and Lie Algebra of Invariance Transformations of $((-1/2)\partial^2/\partial x^2 - i\partial/\partial t)\psi(x, t) = 0$	289
9.7	Eigenfunctions of the Constants of Motion of a Free-Particle	294
9.8	Dynamical Symmetries of the Schrödinger Equations of a Harmonic Oscillator	297
9.9	Use of the Oscillator Group in Perturbation Calculations	300
9.10	Concluding Observations	301
	Historical Note	302
	Exercises	302
	References	303
10.	Spectrum-Generating Lie Algebras and Groups Admitted by Schrödinger Equations	305
10.1	Lie Algebras That Generate Continuous Spectra	306
10.2	Lie Algebras That Generate Discrete Spectra	308
10.3	Dynamical Groups of N-Dimensional Harmonic Oscillators	309
10.4	Linearization of Energy Spectra by Time Dilatation; Spectrum-Generating Dynamical Group of Rigid Rotators	311
10.5	The Angular Momentum Shift Algebra; Dynamical Group of the Laplace Equation	315
10.6	Dynamical Groups of Systems with Both Discrete and Continuous Spectra	319
10.7	Dynamical Group of the Bound States of Morse Oscillators	320
10.8	Dynamical Group of the Bound States of Hydrogen-Like Atoms	322
10.9	Matrix Representations of Generators and Group Operators	325
10.10	Invariant Scalar Products	327
10.11	Direct-Products: $SO(3) \otimes SO(3)$ and the Coupling of Angular Momenta	328

10.12	Degeneracy Groups of Non-interacting Systems. Completions of Direct-Products	329
10.13	Dynamical Groups of Time-dependent Schrödinger Equations of Compound Systems. Many-Electron Atoms	331
	Appendix A: References Providing Invariance Groups of Schrödinger Equations	334
	Appendix B. Reference to Work Dealing with Dynamical Symmetries in Nuclear Shell Theory	334
	Exercises	334
	References	335
11.	Dynamical Symmetry of Regularized Hydrogen-like Atoms	337
11.1	Position-space Realization of the Dynamical Symmetries	337
11.2	The Momentum-space Representation	344
11.3	The Hyperspherical Harmonics Y_{klm}	349
11.4	Bases Provided by Eigenfunctions of J_{12} , J_{34} , J_{56}	354
	Appendix A. Matrix Elements of $SO(4,2)$ Generators	355
	Appendix B. N-Shift Operators For the Hyperspherical Harmonics	356
	References	359
12.	Uncovering Approximate Dynamical Symmetries. Examples From Atomic and Molecular Physics	361
12.1	Introduction	361
12.2	The Stark Effect; One-Electron Diatomics	362
12.3	Correlation Diagrams and Level Crossings: General Remarks	364
12.4	Coupling $SO(4)_1 \otimes SO(4)_2$ to Produce $SO(4)_{12}$	369
12.5	Coupling $SO(4)_1 \otimes SO(4)_2$ to Produce $SO(4)_{1-2}$	371
12.6	Configuration Mixing in Doubly Excited States of Helium-like Atoms	373
12.7	Configuration Mixing Arising From Interactions Within Valence Shells of Second and Third Row Atoms	375
12.8	Origin of the Period-Doubling Displayed in Periodic Charts	375

12.9	Molecular Orbitals in Momentum Space. The Hyperspherical Basis	377
12.10	The Sturmian Ansatz of Avery, Aquilanti and Goscinski	380
	Exercises	384
	References	385
13.	Rovibronic Systems	389
13.1	Introduction	389
13.2	Algebraic Treatment of Anharmonic Oscillators With a Finite Number of Bound States	389
13.3	$U(2) \otimes U(2)$ Model of Vibron Coupling	394
13.4	Spectrum Generating Groups of Rigid Body Rotations	396
13.5	The $U(4)$ Vibron Model of Rotating Vibrating Diatomics	401
13.6	The $U(4) \otimes U(4)$ Model of Rotating Vibrating Triatomics	403
13.7	Concluding Remarks	406
	Exercises	406
	References	406
14.	Dynamical Symmetry of Maxwell's Equations	409
14.1	The Poincare Symmetry of Maxwell's Equations	410
14.2	The Conformal and Inversion Symmetries of Maxwell's Equations; Their Physical Interpretation	413
14.3	Alteration of Wavelengths and Frequencies by a Special Conformal Transformation: Interpretation of Doppler Shifts in Stellar Spectra	420
	Exercises	429
	References	431
	Index	433