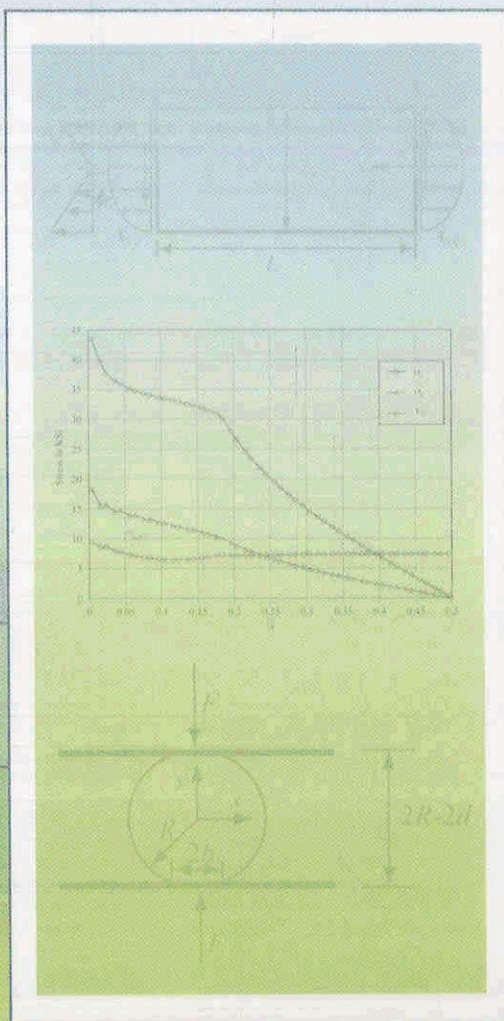


# The Finite Element Method for Mechanics of Solids with ANSYS Applications

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