$$v_i^* \frac{\partial h_i}{\partial x_k} + \sum_{j=1}^{m} u_j^* \frac{\partial g_j}{\partial x_k} = 0$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k$$

$$\Phi_{k+1,j} \le \Phi_k - t_j \beta_k$$

$$\mathbf{x}), r) = f(\mathbf{x}) + \sum_{i=1}^{p} \left[v_i h_i(\mathbf{x}) + \frac{1}{2} r h_i^2(\mathbf{x}) \right.$$

INTRODUCTION TO OPTIMUM DESIGN

THIRD EDITION

JASBIR S. ARORA

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