

Contents

<i>Preface to the second edition</i>	v
<i>Preface</i>	vii
1. An Algebro-Geometric Tool Box	1
1.1 Sheaves	1
1.1.1 Sheaves and Presheaves	1
1.1.2 Sheafification	3
1.1.3 Sheaf Kernel and Cokernel	4
1.2 Schemes	5
1.2.1 Local Ringed Spaces	5
1.2.2 Schemes as Local Ringed Spaces	8
1.2.3 Sheaves over Schemes	9
1.2.4 Topological Properties of Schemes	11
1.3 Projective Schemes	13
1.3.1 Graded Rings	13
1.3.2 Functor Proj	13
1.3.3 Sheaves on Projective Schemes	16
1.4 Categories and Functors	20
1.4.1 Categories	20
1.4.2 Functors	22
1.4.3 Schemes as Functors	23
1.4.4 Abelian Categories	26
1.5 Applications of the Key-Lemma	28
1.5.1 Sheaf of Differential Forms on Schemes	29
1.5.2 Fiber Products	32
1.5.3 Inverse Image of Sheaves	33

1.5.4	Affine Schemes	35
1.5.5	Morphisms into a Projective Space	37
1.6	Group Schemes	38
1.6.1	Group Schemes as Functors	38
1.6.2	Kernel and Cokernel	39
1.6.3	Bialgebras	40
1.6.4	Locally Free Groups	42
1.6.5	Schematic Representations	44
1.7	Cartier Duality	45
1.7.1	Duality of Bialgebras	45
1.7.2	Duality of Locally Free Groups	47
1.8	Quotients by a Group Scheme	50
1.8.1	Naive Quotients	50
1.8.2	Categorical Quotients	52
1.8.3	Geometric Quotients	54
1.9	Morphisms	62
1.9.1	Topological Definitions	62
1.9.2	Diffeo-Geometric Definitions	67
1.9.3	Applications	69
1.10	Cohomology of Coherent Sheaves	73
1.10.1	Coherent Cohomology	73
1.10.2	Summary of Known Facts	77
1.10.3	Cohomological Dimension	78
1.11	Descent	82
1.11.1	Covering Data	82
1.11.2	Descent Data	83
1.11.3	Descent of Schemes	85
1.12	Barsotti–Tate Groups	88
1.12.1	p -Divisible Abelian Sheaf	88
1.12.2	Connected–Étale Exact Sequence	92
1.12.3	Ordinary Barsotti–Tate Group	93
1.13	Formal Scheme	95
1.13.1	Open Subschemes as Functors	96
1.13.2	Examples of Formal Schemes	97
1.13.3	Deformation Functors	101
1.13.4	Connected Formal Groups	102
2.	Elliptic Curves	105
2.1	Curves and Divisors	105

2.1.1	Cartier Divisors	105
2.1.2	Serre–Grothendieck Duality	108
2.1.3	Riemann–Roch Theorem	114
2.1.4	Relative Riemann–Roch Theorem	119
2.2	Elliptic Curves	122
2.2.1	Definition	122
2.2.2	Abel’s Theorem	123
2.2.3	Holomorphic Differentials	125
2.2.4	Taylor Expansion of Differentials	126
2.2.5	Weierstrass Equations of Elliptic Curves	127
2.2.6	Moduli of Weierstrass Type	130
2.3	Geometric Modular Forms of Level 1	134
2.3.1	Functorial Definition	134
2.3.2	Coarse Moduli Scheme	136
2.3.3	Fields of Moduli	138
2.4	Elliptic Curves over $\mathbb{C}$	139
2.4.1	Topological Fundamental Groups	140
2.4.2	Classical Weierstrass Theory	142
2.4.3	Complex Modular Forms	143
2.5	Elliptic Curves over p -Adic Fields	145
2.5.1	Power Series Identities	145
2.5.2	Universal Tate Curves	148
2.5.3	Etale Covering of Tate Curves	153
2.6	Level Structures	155
2.6.1	Isogenies	155
2.6.2	Level N Moduli Problems	157
2.6.3	Generality of Elliptic Curves	163
2.6.4	Proof of Theorem 2.6.8	165
2.6.5	Geometric Modular Forms of Level N	168
2.7	L -Functions of Elliptic Curves	173
2.7.1	L -Functions over Finite Fields	173
2.7.2	Hasse–Weil L -Function	176
2.8	Regularity	180
2.8.1	Regular Rings	180
2.8.2	Regular Moduli Varieties	183
2.9	p -Ordinary Moduli Problems	189
2.9.1	The Hasse Invariant	189
2.9.2	Ordinary Moduli of p -Power Level	193
2.9.3	Irreducibility of p -Ordinary Moduli	195

2.9.4	Moduli Problem of Γ_0 and Γ_1 Type	196
2.9.5	Moduli Problem of $\Gamma_0(p)$ and $\Gamma_1(p)$ Type	198
2.10	Deformation of Elliptic Curves	209
2.10.1	A Theorem of Drinfeld	209
2.10.2	A Theorem of Serre–Tate	211
2.10.3	Deformation of an Ordinary Elliptic Curve	214
3.	Geometric Modular Forms	223
3.1	Integrality	223
3.1.1	Spaces of Modular Forms	223
3.1.2	Horizontal Control Theorem	236
3.2	Vertical Control Theorem	238
3.2.1	False Modular Forms	240
3.2.2	p -Adic Modular Forms	252
3.2.3	Hecke Operators	257
3.2.4	Families of p -Adic Modular Forms	266
3.2.5	Horizontal Control of p -Power Level	271
3.2.6	Control of Hecke algebra	273
3.2.7	Irreducible Components and Analytic Families	275
3.3	Action of $GL(2)$ on Modular Forms	276
3.3.1	Action of $GL_2(\mathbb{Z}/N\mathbb{Z})$	276
3.3.2	Action of $GL_2(\widehat{\mathbb{Z}})$	280
4.	Jacobians and Galois Representations	287
4.1	Jacobians of Stable Curves	287
4.1.1	Non-Singular Curves	287
4.1.2	Union of Two Curves	295
4.1.3	Functorial Properties of Jacobians	298
4.1.4	Self-Duality of Jacobian Schemes	302
4.1.5	Generality on Abelian Schemes	304
4.1.6	Endomorphism of Abelian Schemes	313
4.1.7	ℓ -Adic Galois Representations	318
4.2	Modular Galois Representations	322
4.2.1	Hecke Correspondences	323
4.2.2	Galois Representations on Modular Jacobians	326
4.2.3	Ramification at the Level	330
4.2.4	Ramification of p -Adic Representations at p	335
4.2.5	Modular Galois Representations of Higher Weight	337

4.3	Fullness of Big Galois Representations	342
4.3.1	Big $\mathbb{I}$ -adic Galois Representations	344
4.3.2	Ramification of $\mathbb{I}$ -adic Galois Representations . . .	345
4.3.3	Lie Algebras over p -Adic Ring	346
4.3.4	Lie Algebras of p -Profinite Subgroups of $SL(2)$. .	348
4.3.5	Lie Algebra and Lie Group over $\mathbb{Z}_p$	355
4.3.6	Arithmetic Galois Characters	359
4.3.7	Fullness of Modular Galois Representation	361
4.3.8	Fullness of Elliptic Curves	365
4.3.9	Fullness of Lie Algebra over Λ	368
4.3.10	Fullness of $\mathbb{I}$ -Adic Galois Representation	371
4.3.11	Basic Subgroups	373
4.3.12	Proof of Theorem 4.3.4	380
5.	Modularity Problems	383
5.1	Induced and Extended Galois Representations	384
5.1.1	Induction and Extension	385
5.1.2	Automorphic Induction	392
5.1.3	Artin Representations	395
5.2	Some Other Solutions	402
5.2.1	A Theorem of Wiles	402
5.2.2	Modularity of Extended Galois Representations .	404
5.2.3	Elliptic $\mathbb{Q}$ -Curves	406
5.2.4	Shimura-Taniyama Conjecture	413
5.3	Modularity of Abelian $\mathbb{Q}$ -Varieties	416
5.3.1	Abelian F -varieties of $GL(2)$ -type	417
5.3.2	Endomorphism Algebras of Abelian F -varieties .	424
5.3.3	Application to Abelian $\mathbb{Q}$ -Varieties	425
5.3.4	Abelian Varieties with Real Multiplication	432
	<i>Bibliography</i>	437
	<i>List of Symbols</i>	447
	<i>Statement Index</i>	449
	<i>Index</i>	451