

NEW EDITION OF A CLASSIC BOOK

Time Series Analysis

Forecasting and Control

FOURTH EDITION

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WILEY SERIES IN PROBABILITY AND STATISTICS

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