

# CONTENTS

	page
PREFACE	xi
PRELIMINARIES	1
<b>Part 1. Basic proof theory and computability</b>	
CHAPTER 1. LOGIC	5
1.1. Natural deduction	6
1.2. Normalization	20
1.3. Soundness and completeness for tree models	44
1.4. Soundness and completeness of the classical fragment	52
1.5. Tait calculus	57
1.6. Notes	59
CHAPTER 2. RECURSION THEORY	61
2.1. Register machines	61
2.2. Elementary functions	65
2.3. Kleene's normal form theorem	73
2.4. Recursive definitions	78
2.5. Primitive recursion and for-loops	84
2.6. The arithmetical hierarchy	90
2.7. The analytical hierarchy	94
2.8. Recursive type-2 functionals and well-foundedness	98
2.9. Inductive definitions	102
2.10. Notes	110
CHAPTER 3. GÖDEL'S THEOREMS	113
3.1. $I\Delta_0(\exp)$	114
3.2. Gödel numbers	123
3.3. The notion of truth in formal theories	133
3.4. Undecidability and incompleteness	135
3.5. Representability	137
3.6. Unprovability of consistency	141
3.7. Notes	145

## CONTENTS

### **Part 2. Provable recursion in classical systems**

<b>CHAPTER 4. THE PROVABLY RECURSIVE FUNCTIONS OF ARITHMETIC</b>	<b>149</b>
4.1. Primitive recursion and $I\Sigma_1$	151
4.2. $\varepsilon_0$ -recursion in Peano arithmetic	157
4.3. Ordinal bounds for provable recursion in PA	173
4.4. Independence results for PA	185
4.5. Notes	192
<b>CHAPTER 5. ACCESSIBLE RECURSIVE FUNCTIONS, <math>ID_{&lt;\omega}</math> AND <math>\Pi^1_1\text{-CA}_0</math></b>	<b>195</b>
5.1. The subrecursive stumblingblock	195
5.2. Accessible recursive functions	199
5.3. Proof-theoretic characterizations of accessibility	215
5.4. $ID_{<\omega}$ and $\Pi^1_1\text{-CA}_0$	231
5.5. An independence result: extended Kruskal theorem	237
5.6. Notes	245
<b>Part 3. Constructive logic and complexity</b>	
<b>CHAPTER 6. COMPUTABILITY IN HIGHER TYPES</b>	<b>249</b>
6.1. Abstract computability via information systems	249
6.2. Denotational and operational semantics	266
6.3. Normalization	290
6.4. Computable functionals	296
6.5. Total functionals	304
6.6. Notes	309
<b>CHAPTER 7. EXTRACTING COMPUTATIONAL CONTENT FROM PROOFS</b>	<b>313</b>
7.1. A theory of computable functionals	313
7.2. Realizability interpretation	327
7.3. Refined $A$ -translation	352
7.4. Gödel's Dialectica interpretation	367
7.5. Optimal decoration of proofs	380
7.6. Application: Euclid's theorem	388
7.7. Notes	392
<b>CHAPTER 8. LINEAR TWO-SORTED ARITHMETIC</b>	<b>395</b>
8.1. Provably recursive and complexity in EA( $\cdot$ )	397
8.2. A two-sorted variant $T(\cdot)$ of Gödel's $T$	404
8.3. A linear two-sorted variant $LT(\cdot)$ of Gödel's $T$	412
8.4. Two-sorted systems $A(\cdot)$ , $LA(\cdot)$	422
8.5. Notes	428
<b>BIBLIOGRAPHY</b>	<b>431</b>
<b>INDEX</b>	<b>457</b>