Contents

PREFACE 13

ABOUT THE AUTHORS 16

1 FUNDAMENTAL CONCEPTS 17

1.1 Introduction 17
1.2 Historical Background 17
1.3 Outline of Presentation 18
1.4 Stresses and Equilibrium 18
1.5 Boundary Conditions 20
1.6 Strain–Displacement Relations 21
1.7 Stress–Strain Relations 22
Special Cases, 23
1.8 Temperature Effects 24
1.9 Potential Energy and Equilibrium: The Rayleigh–Ritz Method 25
Potential Energy, Π, 25
Rayleigh–Ritz Method, 28
1.10 Galerkin's Method 30
1.11 Saint Venant's Principle 34
1.12 Von Mises Stress 35
1.13 Principle of Superposition 35
1.14 Computer Programs 36
1.15 Conclusion 36
Historical References 36
Problems 37

2 MATRIX ALGEBRA AND GAUSSIAN ELIMINATION 44

2.1 Matrix Algebra 44
Row and Column Vectors, 45

Addition and Subtraction, 45
Multiplication by a Scalar, 45
Matrix Multiplication, 45
Transposition, 46
Differentiation and Integration, 46
Square Matrix, 47
Diagonal Matrix, 47
Identity Matrix, 47
Symmetric Matrix, 48
Upper Triangular Matrix, 48
Determinant of a Matrix, 48
Matrix Inversion, 48
Eigenvalues and Eigenvectors, 49
Positive Definite Matrix, 51
Cholesky Decomposition, 51

2.2 Gaussian Elimination 51
General Algorithm for Gaussian Elimination, 53
Symmetric Matrix, 56
Symmetric Banded Matrices, 56
Solution with Multiple Right Sides, 58
Gaussian Elimination with Column Reduction, 58
Skyline Solution, 60
Frontal Solution, 61

2.3 Conjugate Gradient Method for Equation Solving 61
Conjugate Gradient Algorithm, 62
Input Data/Output, 62

Problems 63
Program Listings, 65

3 ONE-DIMENSIONAL PROBLEMS 67

3.1 Introduction 67

3.2 Finite Element Modeling 68
Element Division, 68
Numbering Scheme, 69

3.3 Shape Functions and Local Coordinates 71

3.4 The Potential-Energy Approach 75
Element Stiffness Matrix, 76
Force Terms, 78

3.5 The Galerkin Approach 80
Element Stiffness, 80
Force Terms, 81

3.6 Assembly of the Global Stiffness Matrix and Load Vector 82

3.7 Properties of **K** 85

3.8 The Finite Element Equations: Treatment of Boundary Conditions 86

Types of Boundary Conditions, 86
Elimination Approach, 87
Penalty Approach, 92
Multipoint Constraints, 98

3.9 Quadratic Shape Functions 101

3.10 Temperature Effects 108

3.11 Problem Modeling and Boundary Conditions 112
Problem in Equilibrium, 112
Symmetry, 113
Two Elements with Same End Displacements, 113
Problem with a Closing Gap, 114
Input Data/Output, 114

Problems 115
Program Listing, 127

4 TRUSSES **133**

4.1 Introduction 133

4.2 Plane Trusses 134
Local and Global Coordinate Systems, 134
Formulas for Calculating ℓ and m, 135
Element Stiffness Matrix, 136
Stress Calculations, 137
Temperature Effects, 142

4.3 Three-Dimensional Trusses 145

4.4 Assembly of Global Stiffness Matrix for the Banded and Skyline Solutions 147
Assembly for Banded Solution, 147
Skyline Assembly, 148

4.5 Problem Modeling and Boundary Conditions 150
Inclined Support in Two Dimensions, 150
Inclined Support in Three Dimensions—Line Constraint, 150
Inclined Support in Three Dimensions—Plane Constraint, 151
Symmetry and Antisymmetry, 152
Input Data/Output, 154

Problems 155
Program Listing, 163

5 BEAMS AND FRAMES **166**

5.1 Introduction 166
Potential-Energy Approach, 167
Galerkin Approach, 168

5.2 Finite Element Formulation 169
Element Stiffness—Direct Approach, 173

5.3 Load Vector 174

5.4 Boundary Considerations 175
5.5 Shear Force and Bending Moment 176
5.6 Beams on Elastic Supports 178
5.7 Plane Frames 179
5.8 Three-Dimensional Frames 185
5.9 Problem Modeling and Boundary Conditions 189
5.10 Some Comments 190
Input Data/Output, 190
Problems 192
Program Listings, 199

6 TWO-DIMENSIONAL PROBLEMS USING CONSTANT STRAIN TRIANGLES 204

6.1 Introduction 204
6.2 Finite Element Modeling 205
6.3 Constant Strain Triangle (CST) 207
Isoparametric Representation, 208
Potential-Energy Approach, 214
Element Stiffness, 214
Force Terms, 215
Integration Formula on a Triangle, 222
Galerkin Approach, 222
Stress Calculations, 224
Temperature Effects, 226
6.4 Problem Modeling and Boundary Conditions 228
Some General Comments on Dividing into Elements, 231
6.5 Patch Test and Convergence 231
Patch Test, 231
6.6 Orthotropic Materials 232
Temperature Effects, 236
Input Data/Output, 238
Problems 241
Program Listing, 254

7 AXISYMMETRIC SOLIDS SUBJECTED TO AXISYMMETRIC LOADING 258

7.1 Introduction 258
7.2 Axisymmetric Formulation 259
7.3 Finite Element Modeling: Triangular Element 261
Potential Energy Approach, 264
Body Force Term, 265
Rotating Flywheel, 265
Surface Traction, 266
Galerkin Approach, 268

Stress Calculations, 271
Temperature Effects, 272

7.4 Problem Modeling and Boundary Conditions 272
Cylinder Subjected to Internal Pressure, 272
Infinite Cylinder, 273
Press Fit on a Rigid Shaft, 273
Press Fit on an Elastic Shaft, 274
Belleville Spring, 275
Thermal Stress Problem, 276
Input Data/Output, 278

Problems 279
Program Listing, 287

8 TWO-DIMENSIONAL ISOPARAMETRIC ELEMENTS AND NUMERICAL INTEGRATION 289

8.1 Introduction 289

8.2 The Four-Node Quadrilateral 289
Shape Functions, 289
Element Stiffness Matrix, 292
Element Force Vectors, 295

8.3 Numerical Integration 295
Two-Dimensional Integrals, 299
Stiffness Integration, 299
Stress Calculations, 300

8.4 Higher Order Elements 302
Nine-Node Quadrilateral, 303
Eight-Node Quadrilateral, 305
Six-Node Triangle, 306
Integration on a Triangle—Symmetric Points, 307
Integration on a Triangle—Degenerate Quadrilateral, 308

8.5 Four-Node Quadrilateral for Axisymmetric Problems 310

8.6 Conjugate Gradient Implementation of the Quadrilateral Element 311

8.7 Concluding Remarks and Convergence 311

8.8 References for Convergence 313
Input Data/Output, 314

Problems 316
Program Listings, 324

9 THREE-DIMENSIONAL PROBLEMS IN STRESS ANALYSIS 328

9.1 Introduction 328

9.2 Finite Element Formulation 329
Element Stiffness, 332
Force Terms, 333

9.3 Stress Calculations 333
9.4 Mesh Preparation 334
9.5 Hexahedral Elements and Higher Order Elements 338
9.6 Problem Modeling 340
9.7 Frontal Method for Finite Element Matrices 342
Connectivity and Prefront Routine, 343
Element Assembly and Consideration of Specified dof, 344
Elimination of Completed dof, 344
Backsubstitution, 345
Consideration of Multipoint Constraints, 345
Input Data/Output, 346
Problems 348
Program Listings, 352

10 SCALAR FIELD PROBLEMS **361**

10.1 Introduction 361
10.2 Steady-State Heat Transfer 362
One-Dimensional Heat Conduction, 363
One-Dimensional Heat Transfer in Thin Fins, 371
Two-Dimensional Steady-State Heat Conduction, 375
Two-Dimensional Fins, 385
Preprocessing for Program HEAT2D, 386
10.3 Torsion 386
Triangular Element, 388
Galerkin Approach, 389
10.4 Potential Flow, Seepage, Electric and Magnetic Fields, and Fluid Flow in Ducts 392
Potential Flow, 392
Seepage, 394
Electrical and Magnetic Field Problems, 395
Fluid Flow in Ducts, 397
Acoustics, 399
Boundary Conditions, 400
One-Dimensional Acoustics, 400
One-Dimensional Axial Vibrations, 402
Two-Dimensional Acoustics, 404
10.5 Conclusion *405*
Input Data/Output, 405
Problems 407
Program Listings, 418

11 DYNAMIC CONSIDERATIONS **424**

11.1 Introduction 424
11.2 Formulation 424
Solid Body with Distributed Mass, 425

11.3 Element Mass Matrices 427
11.4 Evaluation of Eigenvalues and Eigenvectors 432
Properties of Eigenvectors, 433
Eigenvalue–Eigenvector Evaluation, 433
Inverse Iteration Method, 436
Generalized Jacobi Method, 439
Tridiagonalization and Implicit Shift Approach, 443
Bringing Generalized Problem to Standard Form, 443
Tridiagonalization, 444
Implicit Symmetric QR Step with Wilkinson Shift for Diagonalization, 447
11.5 Interfacing with Previous Finite Element Programs and a Program for Determining Critical Speeds of Shafts 448
11.6 Guyan Reduction 449
11.7 Rigid Body Modes 452
11.8 Conclusion 454
Input Data/Output, 454
Problems 456
Program Listings, 462

12 PREPROCESSING AND POSTPROCESSING 469

12.1 Introduction 469
12.2 Mesh Generation 469
Region and Block Representation, 469
Block Corner Nodes, Sides, and Subdivisions, 470
12.3 Postprocessing 477
Deformed Configuration and Mode Shape, 477
Contour Plotting, 478
Nodal Values from Known Constant Element Values for a Triangle, 479
Least-Squares Fit for a Four-Noded Quadrilateral, 481
12.4 Conclusion 482
Input Data/Output, 483
Problems 484
Program Listings, 486

APPENDIX Proof of $dA = \det \mathbf{J}\, d\xi\, d\eta$ 499

BIBLIOGRAPHY 502

ANSWERS TO SELECTED PROBLEMS 506

INDEX 508