Contents

Preface to Fifth Edition xv

Acknowledgements xvii

1 Preliminaries: Computer Strategies 1
1.1 Introduction 1
1.2 Hardware 2
1.3 Memory Management 2
1.4 Vector Processors 3
1.5 Multi-core Processors 3
1.6 Co-processors 4
1.7 Parallel Processors 4
1.8 Applications Software 5
1.8.1 Compilers 5
1.8.2 Arithmetic 6
1.8.3 Conditions 7
1.8.4 Loops 8
1.9 Array Features 9
1.9.1 Dynamic Arrays 9
1.9.2 Broadcasting 9
1.9.3 Constructors 9
1.9.4 Vector Subscripts 10
1.9.5 Array Sections 11
1.9.6 Whole-array Manipulations 11
1.9.7 Intrinsic Procedures for Arrays 11
1.9.8 Modules 12
1.9.9 Subprogram Libraries 13
1.9.10 Structured Programming 15
1.10 Third-party Libraries 17
1.10.1 BLAS Libraries 17
1.10.2 Maths Libraries 17
1.10.3 User Subroutines 18
1.10.4 MPI Libraries 18
1.11 Visualisation 18
1.11.1 Starting ParaView 19
1.11.2 Display Restrained Nodes 20

	1.11.3 Display Applied Loads	21
	1.11.4 Display Deformed Mesh	21
1.12	Conclusions	23
	References	24
2	**Spatial Discretisation by Finite Elements**	**25**
2.1	Introduction	25
2.2	Rod Element	25
	2.2.1 Rod Stiffness Matrix	25
	2.2.2 Rod Mass Element	28
2.3	The Eigenvalue Equation	28
2.4	Beam Element	29
	2.4.1 Beam Element Stiffness Matrix	29
	2.4.2 Beam Element Mass Matrix	31
2.5	Beam with an Axial Force	31
2.6	Beam on an Elastic Foundation	32
2.7	General Remarks on the Discretisation Process	33
2.8	Alternative Derivation of Element Stiffness	33
2.9	Two-dimensional Elements: Plane Stress	35
2.10	Energy Approach and Plane Strain	38
	2.10.1 Thermoelasticity	39
2.11	Plane Element Mass Matrix	40
2.12	Axisymmetric Stress and Strain	40
2.13	Three-dimensional Stress and Strain	42
2.14	Plate Bending Element	44
2.15	Summary of Element Equations for Solids	47
2.16	Flow of Fluids: Navier–Stokes Equations	47
2.17	Simplified Flow Equations	50
	2.17.1 Steady State	51
	2.17.2 Transient State	53
	2.17.3 Convection	53
2.18	Further Coupled Equations: Biot Consolidation	54
2.19	Conclusions	56
	References	56
3	**Programming Finite Element Computations**	**59**
3.1	Introduction	59
3.2	Local Coordinates for Quadrilateral Elements	59
	3.2.1 Numerical Integration for Quadrilaterals	61
	3.2.2 Analytical Integration for Quadrilaterals	63
3.3	Local Coordinates for Triangular Elements	64
	3.3.1 Numerical Integration for Triangles	65
	3.3.2 Analytical Integration for Triangles	65
3.4	Multi-Element Assemblies	66
3.5	'Element-by-Element' Techniques	68
	3.5.1 Conjugate Gradient Method for Linear Equation Systems	68
	3.5.2 Preconditioning	69

3.5.3 *Unsymmetric Systems* 70
3.5.4 *Symmetric Non-Positive Definite Equations* 71
3.5.5 *Eigenvalue Systems* 71
3.6 Incorporation of Boundary Conditions 72
3.6.1 *Convection Boundary Conditions* 74
3.7 Programming using Building Blocks 75
3.7.1 *Black Box Routines* 76
3.7.2 *Special Purpose Routines* 77
3.7.3 *Plane Elastic Analysis using Quadrilateral Elements* 77
3.7.4 *Plane Elastic Analysis using Triangular Elements* 81
3.7.5 *Axisymmetric Strain of Elastic Solids* 82
3.7.6 *Plane Steady Laminar Fluid Flow* 83
3.7.7 *Mass Matrix Formation* 83
3.7.8 *Higher-Order 2D Elements* 84
3.7.9 *Three-Dimensional Elements* 86
3.7.10 *Assembly of Elements* 90
3.8 Solution of Equilibrium Equations 95
3.9 Evaluation of Eigenvalues and Eigenvectors 96
3.9.1 *Jacobi Algorithm* 96
3.9.2 *Lanczos and Arnoldi Algorithms* 98
3.10 Solution of First-Order Time-Dependent Problems 99
3.11 Solution of Coupled Navier–Stokes Problems 103
3.12 Solution of Coupled Transient Problems 104
3.12.1 *Absolute Load Version* 105
3.12.2 *Incremental Load Version* 106
3.13 Solution of Second-Order Time-Dependent Problems 106
3.13.1 *Modal Superposition* 107
3.13.2 *Newmark or Crank–Nicolson Method* 109
3.13.3 *Wilson's Method* 110
3.13.4 *Complex Response* 111
3.13.5 *Explicit Methods and Other Storage-Saving Strategies* 112
References 113

4 Static Equilibrium of Structures 115
4.1 Introduction 115
Program 4.1 One-dimensional analysis of axially loaded elastic rods using 2-node rod elements 116
Program 4.2 Analysis of elastic pin-jointed frames using 2-node rod elements in two or three dimensions 121
Program 4.3 Analysis of elastic beams using 2-node beam elements (elastic foundation optional) 127
Program 4.4 Analysis of elastic rigid-jointed frames using 2-node beam/rod elements in two or three dimensions 133
Program 4.5 Analysis of elastic–plastic beams or frames using 2-node beam or beam/rod elements in one, two or three dimensions 141
Program 4.6 Stability (buckling) analysis of elastic beams using 2-node beam elements (elastic foundation optional) 150

Program 4.7 Analysis of plates using 4-node rectangular plate elements. Homogeneous material with identical elements. Mesh numbered in x- or y-direction 153
4.2 Conclusions 157
4.3 Glossary of Variable Names 157
4.4 Exercises 159
References 168

5 Static Equilibrium of Linear Elastic Solids 169
5.1 Introduction 169
Program 5.1 Plane or axisymmetric strain analysis of a rectangular elastic solid using 3-, 6-, 10- or 15-node right-angled triangles or 4-, 8- or 9-node rectangular quadrilaterals. Mesh numbered in $x(r)$- or $y(z)$-direction 170
Program 5.2 Non-axisymmetric analysis of a rectangular axisymmetric elastic solid using 8-node rectangular quadrilaterals. Mesh numbered in r- or z-direction 184
Program 5.3 Three-dimensional analysis of a cuboidal elastic solid using 8-, 14- or 20-node brick hexahedra. Mesh numbered in xz-planes then in the y-direction 191
Program 5.4 General 2D (plane strain) or 3D analysis of elastic solids. Gravity loading option 196
Program 5.5 Plane or axisymmetric thermoelastic analysis of an elastic solid using 3-, 6-, 10- or 15-node right-angled triangles or 4-, 8- or 9-node rectangular quadrilaterals. Mesh numbered in $x(r)$- or $y(z)$-direction 205
Program 5.6 Three-dimensional strain of a cuboidal elastic solid using 8-, 14- or 20-node brick hexahedra. Mesh numbered in xz-planes then in the y-direction. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver 209
Program 5.7 Three-dimensional strain of a cuboidal elastic solid using 8-, 14- or 20-node brick hexahedra. Mesh numbered in xz-planes then in the y-direction. No global stiffness matrix. Diagonally preconditioned conjugate gradient solver. Optimised maths library, ABAQUS UMAT version 213
5.2 Glossary of Variable Names 221
5.3 Exercises 224
References 232

6 Material Non-linearity 233
6.1 Introduction 233
6.2 Stress–strain Behaviour 235
6.3 Stress Invariants 236
6.4 Failure Criteria 238
6.4.1 Von Mises 238
6.4.2 Mohr–Coulomb and Tresca 239
6.5 Generation of Body Loads 240
6.6 Viscoplasticity 240

6.7 Initial Stress 242
6.8 Corners on the Failure and Potential Surfaces 243
Program 6.1 Plane-strain-bearing capacity analysis of an elastic–plastic (von Mises) material using 8-node rectangular quadrilaterals. Flexible smooth footing. Load control. Viscoplastic strain method 244
Program 6.2 Plane-strain-bearing capacity analysis of an elastic–plastic (von Mises) material using 8-node rectangular quadrilaterals. Flexible smooth footing. Load control. Viscoplastic strain method. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver 250
Program 6.3 Plane-strain-bearing capacity analysis of an elastic–plastic (Mohr–Coulomb) material using 8-node rectangular quadrilaterals. Rigid smooth footing. Displacement control. Viscoplastic strain method 254
Program 6.4 Plane-strain slope stability analysis of an elastic–plastic (Mohr–Coulomb) material using 8-node rectangular quadrilaterals. Gravity loading. Viscoplastic strain method 260
Program 6.5 Plane-strain earth pressure analysis of an elastic–plastic (Mohr–Coulomb) material using 8-node rectangular quadrilaterals. Rigid smooth wall. Initial stress method 265
6.9 Elastoplastic Rate Integration 270
6.9.1 Forward Euler Method 272
6.9.2 Backward Euler Method 274
6.10 Tangent Stiffness Approaches 275
6.10.1 Inconsistent Tangent Matrix 275
6.10.2 Consistent Tangent Matrix 275
6.10.3 Convergence Criterion 276
Program 6.6 Plane-strain-bearing capacity analysis of an elastic–plastic (von Mises) material using 8-node rectangular quadrilaterals. Flexible smooth footing. Load control. Consistent tangent stiffness. Closest point projection method (CPPM) 277
Program 6.7 Plane-strain-bearing capacity analysis of an elastic–plastic (von Mises) material using 8-node rectangular quadrilaterals. Flexible smooth footing. Load control. Consistent tangent stiffness. CPPM. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver 282
Program 6.8 Plane-strain-bearing capacity analysis of an elastic–plastic (von Mises) material using 8-node rectangular quadrilaterals. Flexible smooth footing. Load control. Consistent tangent stiffness. Radial return method (RR) with 'line search' 286
6.11 The Geotechnical Processes of Embanking and Excavation 289
6.11.1 Embanking 289
Program 6.9 Plane-strain construction of an elastic–plastic (Mohr–Coulomb) embankment in layers on a foundation using 8-node quadrilaterals. Viscoplastic strain method 290
6.11.2 Excavation 294

Program 6.10 Plane-strain construction of an elastic–plastic (Mohr–Coulomb) excavation in layers using 8-node quadrilaterals. Viscoplastic strain method 300
6.12 Undrained Analysis 305
Program 6.11 Axisymmetric 'undrained' strain of an elastic–plastic (Mohr–Coulomb) solid using 8-node rectangular quadrilaterals. Viscoplastic strain method 308
Program 6.12 Three-dimensional strain analysis of an elastic–plastic (Mohr–Coulomb) slope using 20-node hexahedra. Gravity loading. Viscoplastic strain method 313
Program 6.13 Three-dimensional strain analysis of an elastic–plastic (Mohr–Coulomb) slope using 20-node hexahedra. Gravity loading. Viscoplastic strain method. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver 319
6.13 Glossary of Variable Names 322
6.14 Exercises 327
References 331

7 Steady State Flow 333
7.1 Introduction 333
Program 7.1 One-dimensional analysis of steady seepage using 2-node line elements 334
Program 7.2 Plane or axisymmetric analysis of steady seepage using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$- or $y(z)$-direction 337
Program 7.3 Analysis of plane free surface flow using 4-node quadrilaterals. 'Analytical' form of element conductivity matrix 344
Program 7.4 General two- (plane) or three-dimensional analysis of steady seepage 351
Program 7.5 General two- (plane) or three-dimensional analysis of steady seepage. No global conductivity matrix assembly. Diagonally preconditioned conjugate gradient solver 355
7.2 Glossary of Variable Names 359
7.3 Exercises 361
References 367

8 Transient Problems: First Order (Uncoupled) 369
8.1 Introduction 369
Program 8.1 One-dimensional transient (consolidation) analysis using 2-node 'line' elements. Implicit time integration using the 'theta' method 370
Program 8.2 One-dimensional transient (consolidation) analysis (settlement and excess pore pressure) using 2-node 'line' elements. Implicit time integration using the 'theta' method 373
Program 8.3 One-dimensional consolidation analysis using 2-node 'line' elements. Explicit time integration. Element by element. Lumped mass 377

Program 8.4 Plane or axisymmetric transient (consolidation) analysis using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$- or $y(z)$-direction. Implicit time integration using the 'theta' method 380

Program 8.5 Plane or axisymmetric transient (consolidation) analysis using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$- or $y(z)$-direction. Implicit time integration using the 'theta' method. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver 388

Program 8.6 Plane or axisymmetric transient (consolidation) analysis using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$- or $y(z)$-direction. Explicit time integration using the 'theta = 0' method 390

Program 8.7 Plane or axisymmetric transient (consolidation) analysis using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$- or $y(z)$-direction. 'theta' method using an element-by-element product algorithm 394

8.2 Comparison of Programs 8.4, 8.5, 8.6 and 8.7 397

Program 8.8 General two- (plane) or three-dimensional transient (consolidation) analysis. Implicit time integration using the 'theta' method 397

Program 8.9 Plane analysis of the diffusion–convection equation using 4-node rectangular quadrilaterals. Implicit time integration using the 'theta' method. Self-adjoint transformation 401

Program 8.10 Plane analysis of the diffusion–convection equation using 4-node rectangular quadrilaterals. Implicit time integration using the 'theta' method. Untransformed solution 405

Program 8.11 Plane or axisymmetric transient thermal conduction analysis using 4-node rectangular quadrilaterals. Implicit time integration using the 'theta' method. Option of convection and flux boundary conditions 410

8.3 Glossary of Variable Names 416

8.4 Exercises 419

References 422

9 Coupled Problems 423

9.1 Introduction 423

Program 9.1 Analysis of the plane steady-state Navier–Stokes equation using 8-node rectangular quadrilaterals for velocities coupled to 4-node rectangular quadrilaterals for pressures. Mesh numbered in x-direction. Freedoms numbered in the order $u - p - v$ 424

Program 9.2 Analysis of the plane steady-state Navier–Stokes equation using 8-node rectangular quadrilaterals for velocities coupled to 4-node rectangular quadrilaterals for pressures. Mesh numbered in x-direction. Freedoms numbered in the order $u - p - v$. Element-by-element solution using BiCGStab(l) with no preconditioning. No global matrix assembly 429

Program 9.3 One-dimensional coupled consolidation analysis of a Biot poroelastic solid using 2-node 'line' elements. Freedoms numbered in the order $v - u_w$ 433
Program 9.4 Plane strain consolidation analysis of a Biot elastic solid using 8-node rectangular quadrilaterals for displacements coupled to 4-node rectangular quadrilaterals for pressures. Freedoms numbered in order $u - v - u_w$. Incremental load version 438
Program 9.5 Plane strain consolidation analysis of a Biot elastic solid using 8-node rectangular quadrilaterals for displacements coupled to 4-node rectangular quadrilaterals for pressures. Freedoms numbered in order $u - v - u_w$. Incremental load version. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver 445
Program 9.6 Plane strain consolidation analysis of a Biot poroelastic–plastic (Mohr–Coulomb) material using 8-node rectangular quadrilaterals for displacements coupled to 4-node rectangular quadrilaterals for pressures. Freedoms numbered in the order $u - v - u_w$. Viscoplastic strain method 448
9.2 Glossary of Variable Names 454
9.3 Exercises 459
References 460

10 Eigenvalue Problems **461**
10.1 Introduction 461
Program 10.1 Eigenvalue analysis of elastic beams using 2-node beam elements. Lumped mass 462
Program 10.2 Eigenvalue analysis of an elastic solid in plane strain using 4- or 8-node rectangular quadrilaterals. Lumped mass. Mesh numbered in y-direction 465
Program 10.3 Eigenvalue analysis of an elastic solid in plane strain using 4-node rectangular quadrilaterals. Lanczos method. Consistent mass. Mesh numbered in y-direction 469
Program 10.4 Eigenvalue analysis of an elastic solid in plane strain using 4-node rectangular quadrilaterals with ARPACK. Lumped mass. Element-by-element formulation. Mesh numbered in y-direction 474
10.2 Glossary of Variable Names 477
10.3 Exercises 480
References 482

11 Forced Vibrations **483**
11.1 Introduction 483
Program 11.1 Forced vibration analysis of elastic beams using 2-node beam elements. Consistent mass. Newmark time stepping 483
Program 11.2 Forced vibration analysis of an elastic solid in plane strain using 4- or 8-node rectangular quadrilaterals. Lumped mass. Mesh numbered in the y-direction. Modal superposition 489

Program 11.3 Forced vibration analysis of an elastic solid in plane strain using rectangular 8-node quadrilaterals. Lumped or consistent mass. Mesh numbered in the y-direction. Implicit time integration using the 'theta' method 493
Program 11.4 Forced vibration analysis of an elastic solid in plane strain using rectangular 8-node quadrilaterals. Lumped or consistent mass. Mesh numbered in the y-direction. Implicit time integration using Wilson's method 498
Program 11.5 Forced vibration of a rectangular elastic solid in plane strain using 8-node quadrilateral elements numbered in the y-direction. Lumped mass, complex response 501
Program 11.6 Forced vibration analysis of an elastic solid in plane strain using uniform size rectangular 4-node quadrilaterals. Mesh numbered in the y-direction. Lumped or consistent mass. Mixed explicit/ implicit time integration 504
Program 11.7 Forced vibration analysis of an elastic solid in plane strain using rectangular 8-node quadrilaterals. Lumped or consistent mass. Mesh numbered in the y-direction. Implicit time integration using the 'theta' method. No global matrix assembly. Diagonally preconditioned conjugate gradient solver 508
Program 11.8 Forced vibration analysis of an elastic–plastic (von Mises) solid in plane strain using rectangular 8-node quadrilateral elements. Lumped mass. Mesh numbered in the y-direction. Explicit time integration 512
11.2 Glossary of Variable Names 517
11.3 Exercises 521
References 522

12 Parallel Processing of Finite Element Analyses 523
12.1 Introduction 523
12.2 Differences between Parallel and Serial Programs 525
12.2.1 Parallel Libraries 525
12.2.2 Global Variables 526
12.2.3 MPI Library Routines 526
12.2.4 The `_pp` *Appendage* 527
12.2.5 Simple Test Problems 527
12.2.6 Reading and Writing 530
12.2.7 `rest` *Instead of* `nf` 532
12.2.8 Gathering and Scattering 533
12.2.9 Reindexing 533
12.2.10 Domain Composition 533
12.2.11 Third-party Mesh-partitioning Tools 534
12.2.12 Load Balancing 535
Program 12.1 Three-dimensional analysis of an elastic solid. Compare Program 5.6 536
Program 12.2 Three-dimensional analysis of an elastoplastic (Mohr–Coulomb) solid. Compare Program 6.13 542
Program 12.3 Three-dimensional Laplacian flow. Compare Program 7.5 548

Program 12.4 Three-dimensional transient heat conduction–implicit analysis in time. Compare Program 8.5 553
Program 12.5 Three-dimensional transient flow–explicit analysis in time. Compare Program 8.6 562
Program 12.6 Three-dimensional steady-state Navier–Stokes analysis. Compare Program 9.2 565
Program 12.7 Three-dimensional analysis of Biot poro elastic solid. Incremental version. Compare Program 9.5 572
Program 12.8 Eigenvalue analysis of three-dimensional elastic solid. Compare Program 103 576
Program 12.9 Forced vibration analysis of a three-dimensional elastic solid. Implicit integration in time. Compare Program 11.7 581
Program 12.10 Forced vibration analysis of three-dimensional elasto plastic solid. Explicit integration in time. Compare Program 11.8 585
12.3 Graphics Processing Units 589
Program 12.11 Three-dimensional strain of an elastic solid using 8-, 14- or 20-node brick hexahedra. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver. GPU version. Compare Program 5.7 589
12.4 Cloud Computing 594
12.5 Conclusions 596
12.6 Glossary of Variable Names 597
References 602

Appendix A Equivalent Nodal Loads 605

Appendix B Shape Functions and Element Node Numbering 611

Appendix C Plastic Stress-Strain Matrices and Plastic Potential Derivatives 619

Appendix D `main` Library Subprograms 623

Appendix E `geom` Library Subroutines 635

Appendix F Parallel Library Subroutines 639

Appendix G External Subprograms 645

Author Index 649

Subject Index 653