

Contents

1	Coin Tossing	1
1.1	Coin Tossing Probability Model	1
1.2	Combinatorics: $\binom{n}{k}$	3
1.2.1	Multinomial Formula	6
1.2.2	Occupancy Problems* ¹	8
1.3	Random Variables on Ω_n	10
1.4	Binomial Probability Distribution	12
1.5	Weak Law of Large Numbers for Coin Tossing	17
1.6	Consistency Between Probability Spaces	18
1.7	Random Walks: Reflection Principle*	20
1.7.1	The Reflection Principle	22
1.8	Random Walks: First Returns and the Main Theorem*	25
1.8.1	First Returns	29
1.8.2	Main Theorem	32
1.9	Path Maximums and First Visits*	36
1.10	Random Walks: Last Visits and Long Leads*	38
1.10.1	Inverse Sine Approximation for $\alpha_{2k,2n}$	40
1.10.2	Long Leads	43
1.10.3	Experimental Illustration	47
	Appendix: Venn Diagrams	50
	Problems	51
2	Countable Sample Spaces	61
2.1	Countable and Uncountable Sets	61
2.2	Probability for Countable Sample Spaces	65
2.2.1	Basic Properties of a Probability Space	70
2.3	Examples: Finite Sample Spaces	71
2.3.1	Biased Coins and the Binomial Distribution	76
2.3.2	Multinomial Probability Distribution	79
2.4	Examples: Countably Infinite Sample Spaces	81
2.4.1	Poisson Probability Distribution	87
	Problems	95
3	Conditional Probability in Countable Sample Spaces	105

¹Sections marked with an asterisk may be skipped without loss of continuity.

3.1	Conditional Probability	105
3.2	Examples: Conditional Probability	110
3.3	Independence	123
3.4	Coin Tossing and the SLLN	129
	Problems	133
4	Uncountable Sample Spaces	151
4.1	Integration and Probability Theory	153
4.1.1	The Borel Subsets $\beta_{(0,1]}$	154
4.1.2	Properties of $\beta_{(0,1]}$	155
4.1.3	The Probability Space $((0, 1], \beta_{(0,1]}, P)$	155
4.2	Modeling Coin Tossing on $((0, 1], \beta_{(0,1]}, P)$	158
4.2.1	Dyadic Intervals	162
4.2.2	Binomial Distribution	165
4.2.3	Independence of the Coin Tossing Random Variables	166
4.3	Independent Uniform RVs on $((0, 1], \beta_{(0,1]}, P)$	168
4.3.1	U_{o2} and U_{e2}	168
4.3.2	U_{o3} and U_{e3}	175
4.3.3	General Procedure for U_{on} and U_{en}	180
4.4	Formal Definition of Probability Space	183
4.4.1	Basic Properties of a Probability Space	188
	Appendix: The Lebesgue Integral	192
	Appendix: U_o and U_e Are Independent and Uniform	198
	Appendix: The SLLN and Uncountable Sets	201
	Problems	203
5	Continuous Random Variables	213
5.1	Induced Probability Distributions	213
5.2	Independent Random Variables	216
5.2.1	Independent Normal Random Variables	218
5.2.2	Independent Bernoulli Random Variables	222
5.2.3	Independent Poisson Random Variables	224
5.2.4	RVs with a Specified Probability Distribution*	225
5.3	Cumulative Distribution Functions	227
5.4	Example: Random Arrival Times	232
	Appendix - Proof of the Properties of CDFs	235
	Problems	237
6	Expectation	245
6.1	Expectation of Continuous Random Variables	245
6.2	Expectation of the Product of Independent RVs	260
	Problems	264
7	Modeling Random Phenomena	267
7.1	Infinite Sequence of Independent Random Variables	267

7.2	Random Variables and Physical Phenomena	270
7.2.1	One-Dimensional Navigation System	270
7.2.2	Waiting Times	274
7.2.3	Bernoulli Random Process and Arrival Times	278
7.2.4	Binary Communication System	285
7.2.5	Statistics	287
7.2.6	Random Walks and Gambler's Ruin*	289
7.3	The SLLN and Relative Frequency	300
	Problems	304
8	Functions of One Random Variable and Transforms	321
8.1	Functions of One Random Variable	321
8.2	Inequalities	330
8.3	Transforms	335
8.4	Random Walks and Generating Functions*	347
	Problems	354
9	Functions of Two Random Variables	365
9.1	Joint Distributions	365
9.2	Expectation	373
9.3	Independence	377
9.4	One Function of Two Random Variables	381
9.4.1	Sum of Two Random Variables	381
9.4.2	More Examples	392
9.5	Poisson Process	403
9.6	Central Limit Theorem	407
	Appendix - Leibniz Rule for Differentiation	420
	Problems	421
10	Two Functions of Two Random Variables	431
10.1	Transformation of Two Functions of Two Random Variables	431
10.1.1	General Affine Transformation	439
10.1.2	Affine Transformation of Jointly Normal RVs	441
10.2	Covariance and Correlation	447
10.3	Moments and Transforms of Joint Distributions	455
	Appendix: Change of Variables for Multiple Integrals	460
	Problems	463
11	Conditional Probability for Continuous Random Variables	473
11.1	Conditional Probability Density Functions	473
11.2	Application: Binary Transmitter	483
11.3	More Examples of Conditioning	488
11.4	Mean Square Estimation	496
11.4.1	The Conditional Mean as the Best MS Estimate	498
11.5	Application: Navigation System with GPS	502

11.6	Orthogonality Principle and MS Estimation	509
11.6.1	Orthogonality Property of Conditional Expectation	510
11.6.2	Linear MS Estimation and Orthogonality	513
11.7	Random Sums*	524
11.7.1	Probability Density Function of Random Sums	525
11.7.2	Direct Computation of the Mean and Variance	526
11.7.3	Mean and Variance of Random Sums	528
11.7.4	Moment Generating Functions of Random Sums	530
	Problems	532
12	Random Vectors	549
12.1	Joint Densities, Conditional Joint Densities, and Covariances	549
12.2	Definition of a Normal Random Vector	559
12.2.1	Linear Transformations of Normal Random Vectors	561
12.2.2	Derivation of the Normal Density Function	562
12.2.3	Uncorrelated Normal Random Vectors	567
12.3	Conditional Density Functions of Jointly Normal RVs	569
12.4	Linear Mean Square Estimation	574
	Problems	581
13	Bernoulli, Geometric, and Poisson Processes	587
13.1	Bernoulli and Geometric Random Processes	587
13.1.1	Nonstationary Interarrival Waiting Time*	597
13.1.2	Fresh Start Property of Geometric RVs*	601
13.2	Discrete Time Poisson Process	604
13.3	The Poisson Process	607
13.4	Order Statistics	618
13.5	Shot Noise	623
	Problems	629
14	Brownian Motion and White Noise	645
14.1	Random Walks in $\mathbb{R}$	645
14.2	Brownian Motion and the Wiener Process	648
14.2.1	Fokker–Planck Equation	653
14.2.2	Wiener Process from Random Sums*	655
14.3	Gaussian White Noise	658
14.4	Thermal Gaussian White Noise	662
14.5	Discrete-Time Gaussian White Noise	668
14.5.1	Discrete Time to Continuous Time	672
	Appendix: The Dirac Delta Function	677
	Problems	700
15	Stationary Random Processes	703
15.1	Stationarity	703
15.2	Overview of LTI Discrete-Time Systems	708

15.3 Stationarity and Discrete-Time Linear Systems	719
15.4 PSD of Discrete-Time Stationary Processes	724
15.5 Overview of LTI Continuous-Time Systems	737
15.6 Stationarity and Continuous-Time Linear Systems	744
15.7 PSD of Continuous-Time Stationary Processes	747
15.8 Thermal White Gaussian Noise	757
15.9 Complex-Valued Stationary Processes	763
Appendix - Low-Pass Filters with Linear Phase	765
Problems	770
16 Convergence of Random Variables	777
16.1 Weak and Strong Law of Large Numbers	777
16.2 Central Limit Theorem	783
16.3 Types of Convergence	785
16.4 Borel–Cantelli Lemma	805
16.5 Construction of Brownian Motion	808
16.5.1 Complete Orthonormal Functions	808
16.5.2 Construction of Brownian Motion on $[0,1]$	818
16.5.3 Construction of Brownian Motion on $[0, \infty]$	826
16.5.4 Uniform Convergence of the Lévy–Ciesielski Construction	827
Problems	832
17 Statistics	839
17.1 Gamma Distribution	842
17.2 Variance of a Normal Random Variable	846
17.3 Mean of a Normal Random Variable	853
17.4 Maximum Likelihood Estimation	859
17.5 Sufficient Statistics	874
17.6 Goodness-of-Fit Test for Distributions	890
17.6.1 Multinomial Distribution	892
17.6.2 Probability Distribution of $\sum_{i=1}^r \left(\frac{N_i - np_i}{\sqrt{np_i}} \right)^2$	894
Problems	903
18 Kalman Filter	905
18.1 Model	905
18.2 Innovations Approach to the Kalman Filter	909
18.2.1 Innovations Process	910
18.2.2 Kalman Filter via the Innovations Process	912
18.3 Examples	917
18.4 Optimal Linear Mean Square Estimator	921
18.5 The Extended Kalman Filter	923
18.6 Aided Inertial Navigation Using GPS	924
Problems	930

Further Reading	933
Table of Common Distributions	935
References	941
Index	946