Contents

Preface xiii

1. Dimensional Analysis and One-Dimensional Dynamics 1
1.1 Dimensional Analysis 2
1.1.1 The Program of Applied Mathematics 2
1.1.2 Dimensional Methods 5
1.1.3 The Pi Theorem 10
1.1.4 Proof of the Pi Theorem 22
1.2 Scaling 30
1.2.1 Characteristic Scales 30
1.2.2 A Chemical Reactor Problem 33
1.2.3 The Projectile Problem 36
1.3 Differential Equations 46
1.3.1 Review of Elementary Methods 47
1.3.2 Stability and Bifurcation 58

2. Two-Dimensional Dynamical Systems 77
2.1 Phase Plane Phenomena 77
2.2 Linear Systems 87
2.3 Nonlinear Systems 94
2.4 Bifurcations 103
2.5 Reaction Kinetics 112
2.5.1 The Law of Mass Action 113
2.5.2 Enzyme Kinetics 123
2.6 Pathogens 126
2.6.1 Virus Infections 127

2.6.2 Immune System Response 130
2.6.3 Epidemics in Populations 133
2.6.4 Macroparasitic Infections 139

3. Perturbation Methods and Asymptotic Expansions 149
3.1 Regular Perturbation 150
3.1.1 Motion in a Resistive Medium 153
3.1.2 Nonlinear Oscillations 155
3.1.3 The Poincaré–Lindstedt Method 158
3.1.4 Asymptotic Analysis 160
3.2 Singular Perturbation 170
3.2.1 Algebraic Equations 170
3.2.2 Differential Equations 173
3.2.3 Boundary Layers 174
3.3 Boundary Layer Analysis 179
3.3.1 Inner and Outer Approximations 179
3.3.2 Matching 181
3.3.3 Uniform Approximations 183
3.3.4 General Procedures 186
3.4 Initial Layers 191
3.4.1 Damped Spring–Mass System 191
3.4.2 Enzyme Kinetics 195
3.5 The WKB Approximation 202
3.5.1 The Nonoscillatory Case 205
3.5.2 The Oscillatory Case 207
3.6 Asymptotic Expansion of Integrals 210
3.6.1 Laplace Integrals 211
3.6.2 Integration by Parts 214
3.6.3 Other Integrals 216

4. Calculus of Variations 221
4.1 Variational Problems 221
4.1.1 Functionals 221
4.1.2 Examples 223
4.2 Necessary Conditions for Extrema 227
4.2.1 Normed Linear Spaces 227
4.2.2 Derivatives of Functionals 231
4.2.3 Necessary Conditions 233
4.3 The Simplest Problem 236
4.3.1 The Euler Equation 236
4.3.2 Solved Examples 239
4.3.3 First Integrals 240

4.4 Generalizations 245
4.4.1 Higher Derivatives 245
4.4.2 Several Functions 247
4.4.3 Natural Boundary Conditions 249
4.5 Hamilton's Principle 253
4.5.1 Hamilton's Equations 259
4.5.2 The Inverse Problem 262
4.6 Isoperimetric Problems 266

5. Boundary Value Problems and Integral Equations 275
5.1 Boundary-Value Problems 277
5.2 Sturm–Liouville Problems 284
5.2.1 The Eigenvalue Problem 285
5.2.2 Eigenfunction Expansions and Bases 295
5.2.3 Best Approximation and Hilbert Spaces 302
5.3 Classical Fourier Series 310
5.4 Integral Equations 317
5.4.1 Volterra Equations 319
5.4.2 Fredholm Equations with Degenerate Kernels 325
5.4.3 Symmetric Kernels 331
5.5 Green's Functions 339
5.5.1 Inverses of Differential Operators 340
5.5.2 Physical Interpretation 342
5.5.3 Green's Function via Eigenfunctions 348
5.6 Distributions 352
5.6.1 Test Functions 352
5.6.2 Distributions 355
5.6.3 Distribution Solutions to Differential Equations 360

6. Partial Differential Equations 365
6.1 Basic Concepts 365
6.1.1 Linearity and Superposition 370
6.2 Conservation Laws 375
6.2.1 One Dimension 375
6.2.2 Several Dimensions 378
6.2.3 Constitutive Relations 383
6.2.4 Probability and Diffusion 387
6.2.5 Boundary Conditions 390
6.3 Equilibrium Equations 397
6.3.1 Laplace's Equation 397
6.3.2 Basic Properties 401
6.4 Eigenfunction Expansions 404

6.4.1 Spectrum of the Laplacian . . . 405
6.4.2 Evolution Problems . . . 408
6.5 Integral Transforms . . . 415
6.5.1 Laplace Transforms . . . 415
6.5.2 Fourier Transforms . . . 423
6.6 Stability of Solutions . . . 435
6.6.1 Reaction–Diffusion Equations . . . 435
6.6.2 Pattern Formation . . . 437
6.7 Distributions . . . 443
6.7.1 Elliptic Problems . . . 443
6.7.2 Fourier Transforms of Distributions . . . 448
6.7.3 Diffusion Problems . . . 449

7. Wave Phenomena . . . 457
7.1 Waves . . . 457
7.1.1 The Advection Equation . . . 463
7.2 Nonlinear Waves . . . 470
7.2.1 Nonlinear Advection . . . 470
7.2.2 Traveling Wave Solutions . . . 477
7.2.3 Conservation Laws . . . 483
7.3 Quasi-linear Equations . . . 488
7.3.1 Age-Structured Populations . . . 492
7.4 The Wave Equation . . . 497
7.4.1 The Acoustic Approximation . . . 497
7.4.2 Solutions to the Wave Equation . . . 501
7.4.3 Scattering and Inverse Problems . . . 507
7.4.4 The Schrödinger Equation . . . 510

8. Mathematical Models of Continua . . . 523
8.1 Kinematics and Mass Conservation . . . 524
8.1.1 Description of Flow . . . 524
8.1.2 Mass Conservation . . . 530
8.2 Momentum and Energy . . . 534
8.2.1 Momentum Conservation . . . 534
8.2.2 Stress Waves in Solids . . . 538
8.2.3 Thermodynamics and Energy Conservation . . . 545
8.3 Gas Dynamics . . . 551
8.3.1 Riemann's Method . . . 551
8.3.2 Rankine–Hugoniot Conditions . . . 557
8.4 Fluid Motions in $\mathbb{R}^3$ . . . 560
8.4.1 Kinematics . . . 560
8.4.2 Dynamics . . . 567

8.4.3 Energy and Constitutive Theory 574

9. Discrete Models 585
9.1 One-Dimensional Models 586
9.1.1 Linear and Nonlinear Models 586
9.1.2 Equilibria, Stability, and Chaos 591
9.2 Systems of Difference Equations 599
9.2.1 Linear Models 599
9.2.2 Nonlinear Interactions 610
9.3 Stochastic Models 619
9.3.1 Elementary Probability 619
9.3.2 Stochastic Processes 626
9.3.3 Environmental and Demographic Models 630
9.4 Probability-Based Models 636
9.4.1 Markov Processes 636
9.4.2 Random Walks 642
9.4.3 The Poisson Process 647

Index 653