

# Contents

<i>Preface</i>	v
<i>Notation</i>	ix
1. Mathematical structures	1
1.1 Classifying mathematical concepts . . . . .	1
1.2 Defining mathematical structures and mappings . . . . .	2
2. Abstract algebra	5
2.1 Generalizing numbers . . . . .	5
2.1.1 Groups . . . . .	6
2.1.2 Rings . . . . .	8
2.2 Generalizing vectors . . . . .	8
2.2.1 Inner products of vectors . . . . .	10
2.2.2 Norms of vectors . . . . .	11
2.2.3 Multilinear forms on vectors . . . . .	12
2.2.4 Orthogonality of vectors . . . . .	14
2.2.5 Algebras: multiplication of vectors . . . . .	15
2.2.6 Division algebras . . . . .	16
2.3 Combining algebraic objects . . . . .	17
2.3.1 The direct product and direct sum . . . . .	18
2.3.2 The free product . . . . .	19
2.3.3 The tensor product . . . . .	20
2.4 Dividing algebraic objects . . . . .	22
2.4.1 Quotient groups . . . . .	22
2.4.2 Semidirect products . . . . .	23
2.4.3 Quotient rings . . . . .	24

2.4.4	Related constructions and facts . . . . .	25
2.5	Summary . . . . .	25
3.	Vector algebras . . . . .	27
3.1	Constructing algebras from a vector space . . . . .	27
3.1.1	The tensor algebra . . . . .	27
3.1.2	The exterior algebra . . . . .	28
3.1.3	Combinatorial notations . . . . .	30
3.1.4	The Hodge star . . . . .	32
3.1.5	Graded algebras . . . . .	34
3.1.6	Clifford algebras . . . . .	34
3.1.7	Geometric algebra . . . . .	36
3.2	Tensor algebras on the dual space . . . . .	38
3.2.1	The structure of the dual space . . . . .	38
3.2.2	Tensors . . . . .	40
3.2.3	Tensors as multilinear mappings . . . . .	40
3.2.4	Abstract index notation . . . . .	41
3.2.5	Tensors as multi-dimensional arrays . . . . .	43
3.3	Exterior forms . . . . .	44
3.3.1	Exterior forms as multilinear mappings . . . . .	44
3.3.2	Exterior forms as completely anti-symmetric tensors . . . . .	45
3.3.3	Exterior forms as anti-symmetric arrays . . . . .	46
3.3.4	The Clifford algebra of the dual space . . . . .	46
3.3.5	Algebra-valued exterior forms . . . . .	47
3.3.6	Related constructions and facts . . . . .	49
4.	Topological spaces . . . . .	51
4.1	Generalizing surfaces . . . . .	51
4.1.1	Spaces . . . . .	52
4.1.2	Generalizing dimension . . . . .	52
4.1.3	Generalizing tangent vectors . . . . .	53
4.1.4	Existence and uniqueness of additional structure	53
4.1.5	Summary . . . . .	54
4.2	Generalizing shapes . . . . .	55
4.2.1	Defining spaces . . . . .	56
4.2.2	Mapping spaces . . . . .	57
4.3	Constructing spaces . . . . .	60

4.3.1	Cell complexes . . . . .	60
4.3.2	Projective spaces . . . . .	61
4.3.3	Combining spaces . . . . .	62
4.3.4	Classifying spaces . . . . .	64
5.	Algebraic topology . . . . .	67
5.1	Constructing surfaces within a space . . . . .	68
5.1.1	Simplices . . . . .	68
5.1.2	Triangulations . . . . .	69
5.1.3	Orientability . . . . .	70
5.1.4	Chain complexes . . . . .	71
5.2	Counting holes that aren't boundaries . . . . .	71
5.2.1	The homology groups . . . . .	71
5.2.2	Examples . . . . .	73
5.2.3	Calculating homology groups . . . . .	75
5.2.4	Related constructions and facts . . . . .	75
5.3	Counting the ways a sphere maps to a space . . . . .	76
5.3.1	The fundamental group . . . . .	77
5.3.2	The higher homotopy groups . . . . .	79
5.3.3	Calculating the fundamental group . . . . .	80
5.3.4	Calculating the higher homotopy groups . . . . .	80
5.3.5	Related constructions and facts . . . . .	80
6.	Manifolds . . . . .	83
6.1	Defining coordinates and tangents . . . . .	84
6.1.1	Coordinates . . . . .	84
6.1.2	Tangent vectors and differential forms . . . . .	85
6.1.3	Frames . . . . .	89
6.1.4	Tangent vectors in terms of frames . . . . .	91
6.2	Mapping manifolds . . . . .	92
6.2.1	Diffeomorphisms . . . . .	92
6.2.2	The differential and pullback . . . . .	92
6.2.3	Immersions and embeddings . . . . .	94
6.2.4	Critical points . . . . .	95
6.3	Derivatives on manifolds . . . . .	96
6.3.1	Derivations . . . . .	96
6.3.2	The Lie derivative of a vector field . . . . .	97
6.3.3	The Lie derivative of an exterior form . . . . .	99

6.3.4	The exterior derivative of a 1-form . . . . .	101
6.3.5	The exterior derivative of a k-form . . . . .	104
6.3.6	Relationships between derivations . . . . .	106
6.4	Homology on manifolds . . . . .	107
6.4.1	The Poincaré lemma . . . . .	107
6.4.2	de Rham cohomology . . . . .	108
6.4.3	Poincaré duality . . . . .	109
7.	Lie groups . . . . .	111
7.1	Combining algebra and geometry . . . . .	111
7.1.1	Spaces with multiplication of points . . . . .	111
7.1.2	Vector spaces with topology . . . . .	112
7.2	Lie groups and Lie algebras . . . . .	113
7.2.1	The Lie algebra of a Lie group . . . . .	114
7.2.2	The Lie groups of a Lie algebra . . . . .	115
7.2.3	Relationships between Lie groups and Lie algebras . . . . .	116
7.2.4	The universal cover of a Lie group . . . . .	117
7.3	Matrix groups . . . . .	119
7.3.1	Lie algebras of matrix groups . . . . .	119
7.3.2	Linear algebra . . . . .	120
7.3.3	Matrix groups with real entries . . . . .	122
7.3.4	Other matrix groups . . . . .	123
7.3.5	Manifold properties of matrix groups . . . . .	124
7.3.6	Matrix group terminology in physics . . . . .	126
7.4	Representations . . . . .	127
7.4.1	Group actions . . . . .	128
7.4.2	Group and algebra representations . . . . .	130
7.4.3	Lie group and Lie algebra representations . . . . .	131
7.4.4	Combining and decomposing representations . . . . .	132
7.4.5	Other representations . . . . .	134
7.5	Classification of Lie groups . . . . .	135
7.5.1	Compact Lie groups . . . . .	136
7.5.2	Simple Lie algebras . . . . .	138
7.5.3	Classifying representations . . . . .	140
8.	Clifford groups . . . . .	141
8.1	Classification of Clifford algebras . . . . .	141
8.1.1	Isomorphisms . . . . .	141

8.1.2	Representations and spinors . . . . .	143
8.1.3	Pauli and Dirac matrices . . . . .	145
8.1.4	Chiral decomposition . . . . .	148
8.2	Clifford groups and representations . . . . .	149
8.2.1	Reflections . . . . .	149
8.2.2	Rotations . . . . .	150
8.2.3	Lie group properties . . . . .	152
8.2.4	Lorentz transformations . . . . .	153
8.2.5	Representations in spacetime . . . . .	156
8.2.6	Spacetime and spinors in geometric algebra . . . . .	159
9.	Riemannian manifolds . . . . .	161
9.1	Introducing parallel transport of vectors . . . . .	161
9.1.1	Change of frame . . . . .	161
9.1.2	The parallel transporter . . . . .	162
9.1.3	The covariant derivative . . . . .	163
9.1.4	The connection . . . . .	165
9.1.5	The covariant derivative in terms of the connection	166
9.1.6	The parallel transporter in terms of the connection	169
9.1.7	Geodesics and normal coordinates . . . . .	170
9.1.8	Summary . . . . .	172
9.2	Manifolds with connection . . . . .	175
9.2.1	The covariant derivative on the tensor algebra .	175
9.2.2	The exterior covariant derivative of vector-valued forms . . . . .	177
9.2.3	The exterior covariant derivative of algebra-valued forms . . . . .	179
9.2.4	Torsion . . . . .	181
9.2.5	Curvature . . . . .	184
9.2.6	First Bianchi identity . . . . .	187
9.2.7	Second Bianchi identity . . . . .	190
9.2.8	The holonomy group . . . . .	193
9.3	Introducing lengths and angles . . . . .	194
9.3.1	The Riemannian metric . . . . .	194
9.3.2	The Levi-Civita connection . . . . .	196
9.3.3	Independent quantities and dependencies . . . . .	198
9.3.4	The divergence and conserved quantities . . . . .	199
9.3.5	Ricci and sectional curvature . . . . .	203
9.3.6	Curvature and geodesics . . . . .	206

9.3.7	Jacobi fields and volumes . . . . .	209
9.3.8	Summary . . . . .	212
9.3.9	Related constructions and facts . . . . .	215
10.	Fiber bundles	217
10.1	Gauge theory . . . . .	217
10.1.1	Matter fields and gauges . . . . .	217
10.1.2	The gauge potential and field strength . . . . .	218
10.1.3	Spinor fields . . . . .	219
10.2	Defining bundles . . . . .	222
10.2.1	Fiber bundles . . . . .	222
10.2.2	$G$ -bundles . . . . .	225
10.2.3	Principal bundles . . . . .	226
10.3	Generalizing tangent spaces . . . . .	229
10.3.1	Associated bundles . . . . .	229
10.3.2	Vector bundles . . . . .	230
10.3.3	Frame bundles . . . . .	233
10.3.4	Gauge transformations on frame bundles . . . . .	237
10.3.5	Smooth bundles and jets . . . . .	241
10.3.6	Vertical tangents and horizontal equivariant forms	242
10.4	Generalizing connections . . . . .	246
10.4.1	Connections on bundles . . . . .	246
10.4.2	Parallel transport on the frame bundle . . . . .	247
10.4.3	The exterior covariant derivative on bundles . . . . .	250
10.4.4	Curvature on principal bundles . . . . .	251
10.4.5	The tangent bundle and solder form . . . . .	252
10.4.6	Torsion on the tangent frame bundle . . . . .	256
10.4.7	Spinor bundles . . . . .	257
10.5	Characterizing bundles . . . . .	259
10.5.1	Universal bundles . . . . .	259
10.5.2	Characteristic classes . . . . .	262
10.5.3	Related constructions and facts . . . . .	263
Appendix A	Categories and functors	265
A.1	Generalizing sets and mappings . . . . .	265
A.2	Mapping mappings . . . . .	266
Bibliography		269
Index		271