

DIFFUSIONS, MARKOV PROCESSES AND MARTINGALES

Volume 1
Foundations

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Some Frequently Used Notation

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