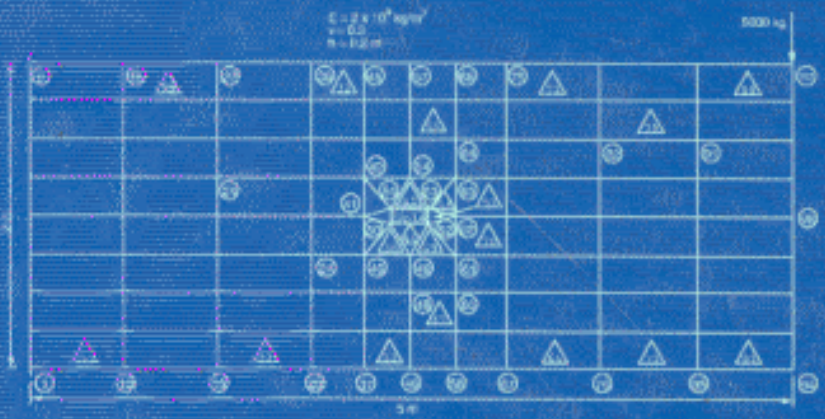
Introductory FINITE ELEMENT METHOD



Chandrakant S. Desai
Tribikram Kundu

Contents

1 Introduction 1
Basic Concept 1
Process of Discretization 3
Subdivision 3
Continuity 4
Convergence 4
Bounds 5
Error 6
Principles and Laws 7
Cause and Effect 10
Important Comment 10
Review Assignments 10
Home Assignment 1 11
Home Assignment 2 12
References 12

2 Steps in the Finite Element Method 13
Introduction 13
General Idea 13
Step 1. Discretize and Select Element Configuration 14
Step 2. Select Approximation Models or Functions 16
Step 3. Define Strain (Gradient)-Displacement (Unknown) and Stress–Strain (Constitutive) Relationships 18
Step 4. Derive Element Equations 19
Energy Methods 19
Stationary Value 20
Potential Energy 20
Method of Weighted Residuals 22
Element Equations 25
Step 5. Assemble Element Equations to Obtain Global or Assemblage Equations and Introduce Boundary Conditions 26
Boundary Conditions 27
Step 6. Solve for the Primary Unknowns 29
Step 7. Solve for Derived or Secondary Quantities 29
Step 8. Interpretation of Results 29
Introduction to Variational Calculus 30
Definitions of Functions and Functionals 30
Variations of Functions 31
Stationary Values of Functions and Functionals 36

Δf_x to Δf Conversion 37
More on the Stationary Value of a Functional — Physical Interpretation 38
Natural and Forced Boundary Conditions 39
Two-Dimensional Problems 41
Summary 47
Problems 48
References 50

3 One-Dimensional Stress Deformation 53
Introduction 53
Step 1. Discretization and Choice of Element Configuration 53
Explanation of Global and Local Coordinates 54
Local and Global Coordinate System for the One-Dimensional Problem 55
Step 2. Select Approximation Model or Function for the Unknown (Displacement) 56
Generalized Coordinates 57
Interpolation Functions 60
Relation between Local and Global Coordinates 61
Variation of Element Properties 61
Requirements for Approximation Functions 62
Step 3. Define Strain-Displacement and Stress–Strain Relations 64
Stress–Strain Relation 65
Step 4. Derive Element Equations 66
Principle of Minimum Potential Energy 66
Functional for One-Dimensional Stress Deformation Problem 69
Total Potential Energy Approach 69
Variational Principle Approach 71
Expansion of Terms 73
Integration 75
Comment 77
Step 5. Assemble Element Equations to Obtain Global Equations 77
Direct Stiffness Method 80
Boundary Conditions 82
Types of Boundary Conditions 83
Homogeneous or Zero-Valued Boundary Condition 83
Nonzero Boundary Conditions 84
Step 6. Solve for Primary Unknowns: Nodal Displacements 85
Step 7. Solve for Secondary Unknowns; Strains and Stresses 88
Strains and Stresses 88
Step 8. Interpretation and Display of Results 90

Formulation by Galerkin's Method 92
Explanation and Relevance of Interpolation Functions 92
Comment 101
Computer Implementation 101
Other Procedures for Formulation 101
Comment 102
Complementary Energy Approach 102
Comment 105
Mixed Approach 105
Variational Method 105
Residual Methods 106
Variational Method 106
Comment 109
Galerkin's Method 109
Comment 110
Bounds 112
Comment 113
Advantages of the Finite Element Method 113
Problems 114
References 120

4 One-Dimensional Flow 123
Theory and Formulation 123
Governing Equation 123
Finite Element Formulation 124
Step 1. Choose Element Configuration 124
Step 2. Choose Approximation Function 124
Step 3. Define Gradient-Potential Relation and Constitutive Law 125
Step 4. Derive Element Equations 127
Variational Approach 127
Evaluation of [k] and {Q} 128
Step 5. Assemble 129
Step 6. Solve for Potentials 129
Step 7. Secondary Quantities 131
Step 8. Interpret and Plot Results 132
Formulation by Galerkin's Method 132
Forced and Natural Boundary Conditions for Flow Problems 133
Problems 135
Bibliography 138

5 One-Dimensional Time Dependent Flow: Introduction to Uncoupled and Coupled Problems 139
Uncoupled Case 139
Initial Stress 142
Residual Stresses 142
Comment 144

Time-Dependent Problems 144
Governing Equation 144
Step 1. Discretize and Choose Element Configuration 146
Step 2. Choose Approximation Model 146
Step 3. Define Gradient-Temperature and Constitutive Relation 147
Step 4. Derive Element Equations 147
Layered Media 151
Solution in Time 151
Step 4. Derivation by Galerkin's Method 153
Step 5. Assembly for Global Equations 155
Boundary Conditions 157
Step 6. Solve for Primary Unknowns 159
Second Time Increment 160
Step 7. Compute the Derived or Secondary Quantities 161
One-Dimensional Consolidation 162
Computer Code 165
Problems 169
References 175

6 Finite Element Codes: One- and Two-Dimensional Problems 177
One-Dimensional Code 177
Deformation, Flow, and Temperature/Consolidation Problems 177
Philosophy of Codes 178
Stages 178
Stage 1. Input Quantities 178
Stage 2. Initialize 180
Stage 3. Compute Element Matrices 180
Stage 4. Assemble 181
Stage 5. Concentrated Forces 181
Stage 6. Boundary Conditions 181
Stage 7. Time Integration 181
Stage 8. Solve Equations 181
Stage 9. Set $\{R\}_t = \{H\} = \{R\}_{t+\Delta t}$ 181
Stage 10. Output Quantities 182
Explanation of Major Symbols and Arrays 182
User's Guide for Code DFT/C-1DFE 184
Two-Dimensional Code 187
Stress Deformation and Field Problems 187
User's Guide for Plane-2D 188
Program Input to be Prepared by User 188
General Comments 188
Conventions 188
Units 188

0. Title Record 188
1. Problem Parameters Record (Input Set 1) 189
2. Material Property Specifications (Input Set 2) 189
3. Nodal Point Specifications (Input Set 3) 189
Automatic Node Generation 190
4. Element Specification (Input Set 4) 190
Automatic Element Generation 190
5. Surface Loading (Pressure) Specifications (Input Set 5) 191
6. Blank Line (required if more than one data set) 192
Sample Problems for Plane-2D 192
Plane Stress 192
User's Guide for Field-2D 195
Sample Problems for FIELD-2D 197
References 198

7 Beam Bending and Beam-Column 199
Introduction 199
Step 1. Discretize and Choose Element Configuration 199
Step 2. Choose Approximation Model 200
Comment on Requirements for Approximation Function 204
Step 3. Define Strain-Displacement and Stress-Strain Relationships 204
Step 4. Derive Element Equations 205
Energy Approach 205
Derivation Using Galerkin's Method 207
Steps 5 to 8 210
Closed Form Solutions 212
Secondary Quantities 212
Comment 215
Mesh Refinement 215
Higher-Order Approximation 216
Beam-Column 219
Step 1 219
Step 2 219
Steps 3 to 5 221
Comment 222
Other Procedures of Formulation 222
Complementary Energy Approach 222
Mixed Approach 224
Problems 228
References 237

8 One-Dimensional Mass Transport 239
Introduction 239
Finite Element Formulation 239

Step 4. Derivation of Element Equations 240
Step 5. Assembly 243
Solution in Time 244
Convection Parameter v_x 244
Comment 245
Comment 247
References 248
Bibliography 248

9 One-Dimensional Stress Wave Propagation 251
Introduction 251
Finite Element Formulation 252
Step 4. Derive Element Equations 252
Step 5. Assemble Element Equations 255
Time Integration 255
Boundary and Initial Conditions 258
Boundary Conditions 261
Damping 261
Problems 263
References 264
Bibliography 264

10 Two- and Three-Dimensional Formulations 267
Introduction 267
Two-Dimensional Formulation 267
Triangular Element 267
Requirements for the Approximation Function 270
Integration of $[B]^T[B]$ 272
Quadrilateral Element 273
Approximation Model for Unknown 273
Requirements for the Approximation Function 276
Secondary Unknowns 276
Integration of $[B]^T[B]$ 280
Numerical Integration 280
Three-Dimensional Formulation 284
Tetrahedron Element 285
Brick Element 286
Problems 288
References 294

11 Torsion 295
Introduction 295
Finite Element Formulation (Displacement Approach) 296
Step 3. Gradient-Unknown Relation and Constitutive Law 297
Step 4. Derive Element Equations 297

Step 5. Assembly 301
Step 6. Secondary Quantities 305
Twisting Moment 307
Comparisons of Numerical Predictions and Closed Form Solutions 309
Twisting Moment 309
Shear Stresses 309
Comment 311
Stress Approach 311
Boundary Conditions 315
Step 6. Secondary Quantities 316
Shear Stresses 316
Twisting Moment 317
Step 8. Interpretation and Plots 318
Comparisons 319
Bounds 320
Comparisons 323
Comparisons 326
Review and Comments 327
Hybrid Approach 328
Step 4. Element Equations 330
Element Stiffness Matrix 335
Inner Elements 336
Computation of Boundary Shear Stresses 337
Assembly 337
Shear Stresses 341
Twisting Moment 342
Mixed Approach 345
Step 4. Derive Element Equations 346
Evaluation of Element Matrices and Load Vector 347
Static Condensation 354
Problems 356
References 360

12 Other Field Problems: Potential, Thermal, Fluid, and Electrical Flow 361
Introduction 361
Potential Flow 362
Boundary Conditions 363
Finite Element Formulation 364
Step 4. Derive Element Equations 365
Derivation of the Governing Equation 366
Step 5. Assembly 369
Evaluation of {Q} 369
Stream Function Formulation 371
Secondary Quantities 372

Thermal or Heat Flow Problem 377
Seepage 380
Electromagnetic Problems 384
Computer Code 384
Problems 385
References 392
Bibliography 393

13 Two-Dimensional Stress-Deformation Analysis 395
Introduction 395
Plane Deformations 395
Plane Stress Idealization 395
Plane Strain Idealization 397
Axisymmetric Idealization 398
Strain-Displacement Relations 399
Finite Element Formulation 400
Requirements for Approximation Function 402
Plane Stress Idealization 402
Step 4. Derive Element Equations 404
Evaluation of [k] and {Q} 405
Triangular Element 408
Comment on Convergence 417
Computer Code 418
Comment 419
Partial Results 422
Comment 424
Comment 427
Problems 428
References 436

14 Multicomponent Systems: Building Frame and Foundation 437
Introduction 437
Various Components 437
Beam-Column 437
Plate or Slab 441
Membrane Effects 441
Bending 441
Assembly 446
Representation of Foundation 446
Computer Code 447
Transformation of Coordinates 455
Problems 457
References 458

Appendix 1 Various Numerical Procedures: Solution of Beam Bending Problem **459**
Introduction 459
Various Residual Procedures 460
Beam Bending by Various Procedures 461
Collocation 463
Subdomain Method 464
Least-Squares Method 465
Galerkin's Method 466
Ritz Method 466
Comment 467
Finite Element Method 467
Finite Difference Method 468
Comparisons of the Methods 469
References 470

Appendix 2 Solution of Simultaneous Equations **471**
Introduction 471
Methods of Solution 472
Gaussian Elimination 472
Back Substitution 474
Banded and Symmetric Systems 474
Solution Procedure 475
Iterative Procedures 477
Comment 481
References 482

Appendix 3 Computer Codes **483**
Introduction 483
Advanced Codes 483
Other Codes on Diffusion-Convection 484
Other Codes on Flow Problems 485
Reference 486

Index **487**