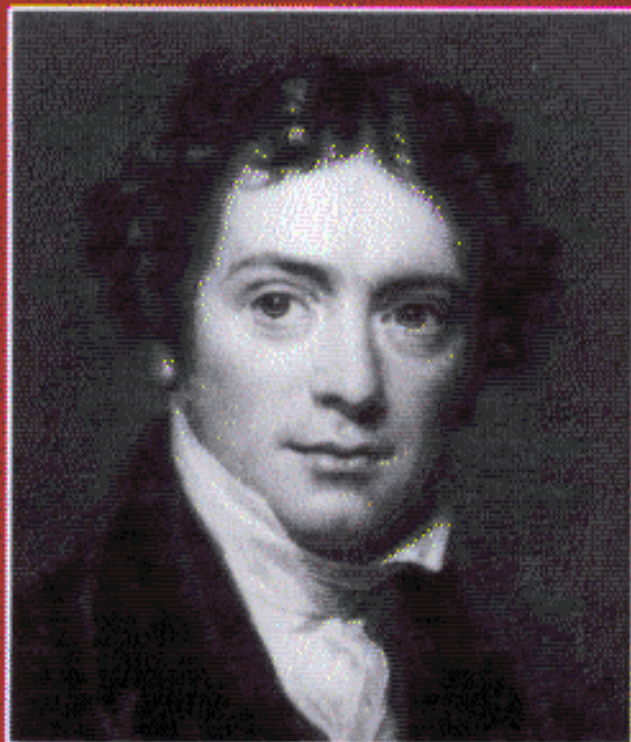


**CAMBRIDGE TEXTS
IN APPLIED
MATHEMATICS**

An Introduction to Magnetohydrodynamics



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