

OTAEM

OXFORD TEXTS IN APPLIED AND ENGINEERING MATHEMATICS

Finite Element
and Boundary Element
Applications in
Quantum Mechanics

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