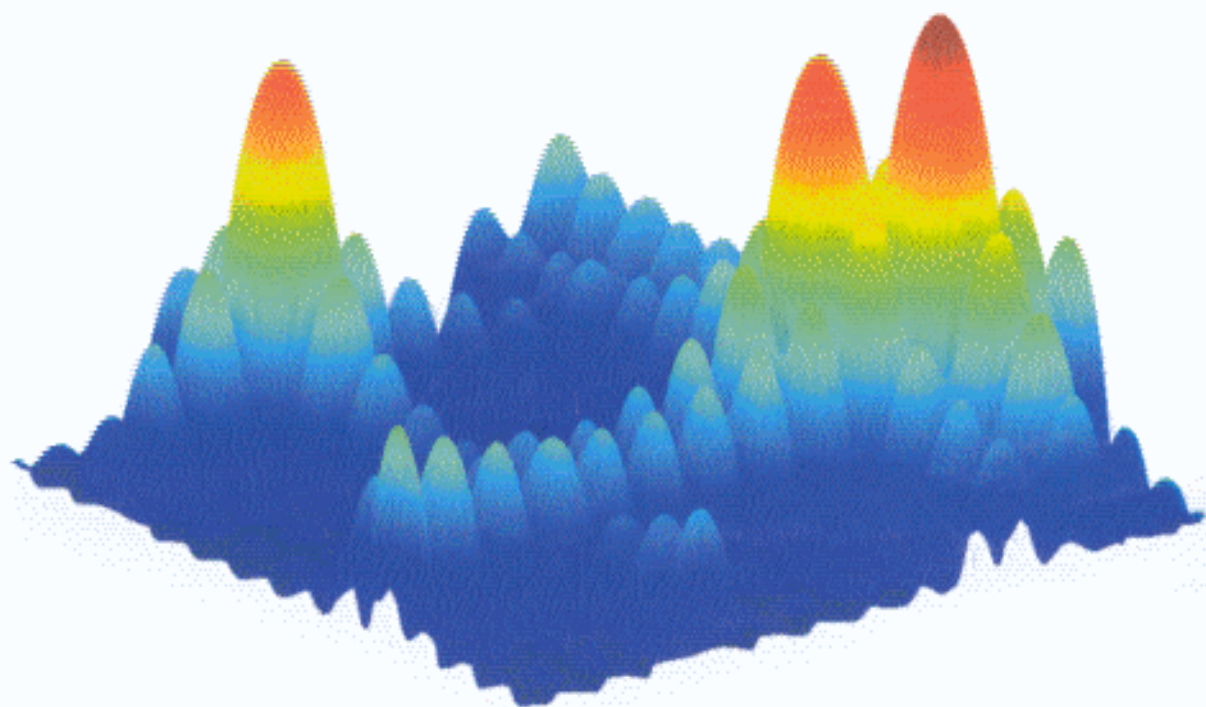




# OPTIMUM ARRAY PROCESSING

Part IV of *Detection, Estimation, and Modulation Theory*

Harry L. Van Trees

# Contents

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