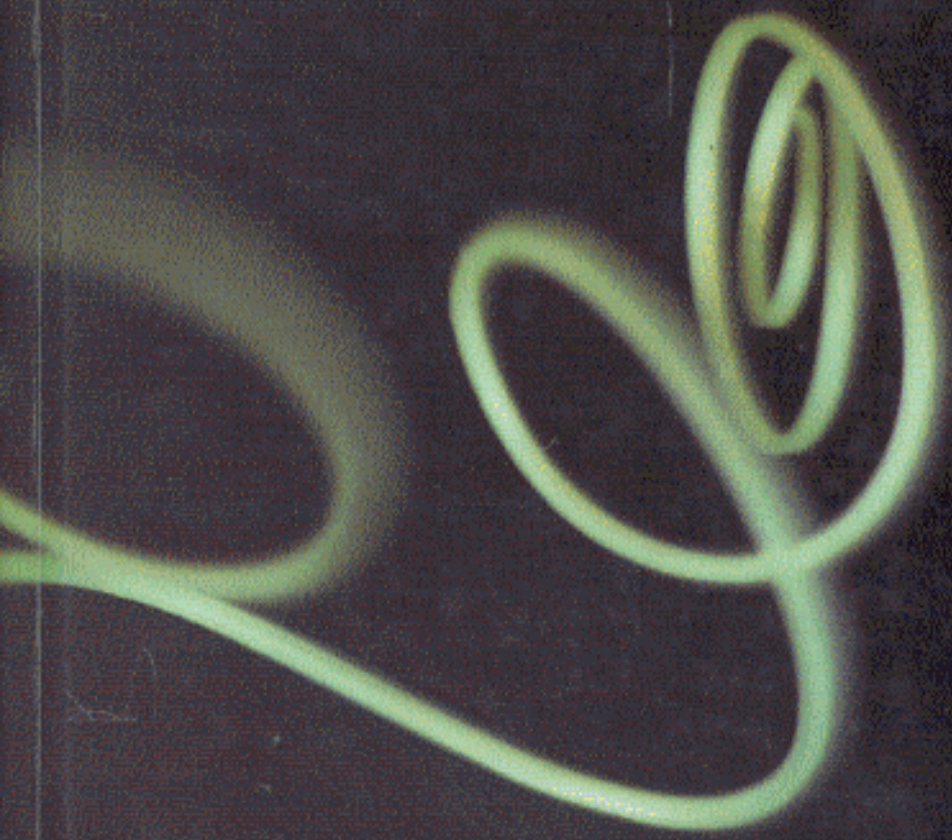
Selwyn Hollis



Differential Equations

with

BOUNDARY VALUE PROBLEMS

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