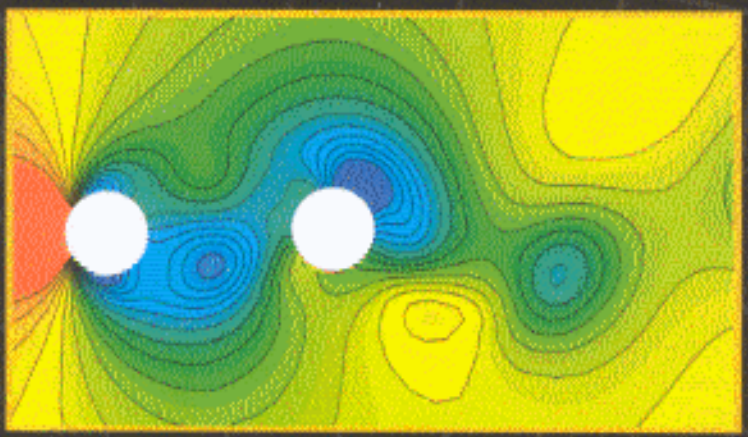
Finite Element Methods for Flow Problems



Jean Donea and Antonio Huerta

WILEY

Contents

Preface *xi*

1 *Introduction and preliminaries* *1*
- *1.1 Finite elements in fluid dynamics* *1*
- *1.2 Subjects covered* *2*
- *1.3 Kinematical descriptions of the flow field* *4*
  - *1.3.1 Lagrangian and Eulerian descriptions* *5*
  - *1.3.2 ALE description of motion* *8*
  - *1.3.3 The fundamental ALE equation* *11*
  - *1.3.4 Time derivative of integrals over moving volumes* *12*
- *1.4 The basic conservation equations* *13*
  - *1.4.1 Mass equation* *13*
  - *1.4.2 Momentum equation* *13*
  - *1.4.3 Internal energy equation* *15*
  - *1.4.4 Total energy equation* *17*
  - *1.4.5 ALE form of the conservation equations* *18*
  - *1.4.6 Closure of the initial boundary value problem* *19*
- *1.5 Basic ingredients of the finite element method* *19*
  - *1.5.1 Mathematical preliminaries* *19*

1.5.2 *Trial solutions and weighting functions* 21
1.5.3 *Compact integral forms* 23
1.5.4 *Strong and weak forms of a boundary value problem* 23
1.5.5 *Finite element spatial discretization* 27

2 *Steady transport problems* 33
2.1 *Problem statement* 33
2.1.1 *Strong form* 33
2.1.2 *Weak form* 35
2.2 *Galerkin approximation* 36
2.2.1 *Piecewise linear approximation in 1D* 37
2.2.2 *Analysis of the discrete equation* 40
2.2.3 *Piecewise quadratic approximation in 1D* 45
2.2.4 *Analysis of the discrete equations* 47
2.3 *Early Petrov–Galerkin methods* 50
2.3.1 *Upwind approximation of the convective term* 50
2.3.2 *First finite elements of upwind type* 51
2.3.3 *The concept of balancing diffusion* 53
2.4 *Stabilization techniques* 59
2.4.1 *The SUPG method* 60
2.4.2 *The Galerkin/Least-squares method* 63
2.4.3 *The stabilization parameter* 64
2.5 *Other stabilization techniques and new trends* 65
2.5.1 *Finite increment calculus* 66
2.5.2 *Bubble functions and wavelet approximations* 67
2.5.3 *The variational multiscale method* 68
2.5.4 *Complements* 70
2.6 *Applications and solved exercises* 70
2.6.1 *Construction of a bubble function method* 70
2.6.2 *One-dimensional transport* 72
2.6.3 *Convection–diffusion across a source term* 74
2.6.4 *Convection–diffusion skew to the mesh* 75
2.6.5 *Convection–diffusion–reaction in 2D* 76
2.6.6 *The Hemker problem* 78

3 *Unsteady convective transport* 79
3.1 *Introduction* 79
3.2 *Problem statement* 81

3.3 The method of characteristics 82
3.3.1 The concept of characteristic lines 82
3.3.2 Properties of the linear convection equation 84
3.3.3 Methods based on the characteristics 87
3.4 Classical time and space discretization techniques 91
3.4.1 Time discretization 92
3.4.2 Galerkin spatial discretization 94
3.5 Stability and accuracy analysis 98
3.5.1 Analysis of stability by Fourier techniques 99
3.5.2 Analysis of classical time-stepping schemes 101
3.5.3 The modified equation method 105
3.6 Taylor–Galerkin Methods 107
3.6.1 The need for higher-order time schemes 107
3.6.2 Third-order explicit Taylor–Galerkin method 108
3.6.3 Fourth-order explicit leap-frog method 113
3.6.4 Two-step explicit Taylor–Galerkin methods 114
3.7 An introduction to monotonicity-preserving schemes 117
3.8 Least-squares-based spatial discretization 120
3.8.1 Least-squares approach for the θ family of methods 121
3.8.2 Taylor least-squares method 122
3.9 The discontinuous Galerkin method 124
3.10 Space–time formulations 126
3.10.1 Time-discontinuous Galerkin formulation 126
3.10.2 Time-discontinuous least-squares formulation 128
3.10.3 Space-timeGalerkin/Least-squaresformulation 128
3.11 Applications and solved exercises 129
3.11.1 Propagation of a cosine profile 129
3.11.2 Travelling wave package 133
3.11.3 The rotating cone problem 135
3.11.4 Propagation of a steep front 137

4 Compressible Flow Problems 147
4.1 Introduction 147
4.2 Nonlinear hyperbolic equations 149
4.2.1 Scalar equations 149
4.2.2 Weak solutions and entropy condition 152
4.2.3 Time and space discretization 156
4.3 The Euler equations 159

4.3.1 Strong form of the conservation equations 159
4.3.2 The quasi-linear form of the Euler equations 161
4.3.3 Basic properties of the Euler equations 163
4.3.4 Boundary conditions 165
4.4 Spatial discretization techniques 166
4.4.1 Galerkin formulation 167
4.4.2 Upwind-type discretizations 168
4.5 Numerical treatment of shocks 176
4.5.1 Introduction 176
4.5.2 Early artificial diffusion methods 178
4.5.3 High-resolution methods 180
4.6 Nearly incompressible flows 186
4.7 Fluid–structure interaction 187
4.7.1 Acoustic approximation 189
4.7.2 Nonlinear transient dynamic problems 191
4.7.3 Illustrative examples 196
4.8 Solved exercises 199
4.8.1 One-step Taylor–Galerkin solution of Burgers' equation 199
4.8.2 The shock tube problem 202

5 Unsteady convection–diffusion problems 209
5.1 Introduction 209
5.2 Problem statement 210
5.3 Time discretization procedures 211
5.3.1 Classical methods 211
5.3.2 Fractional-step methods 213
5.3.3 High-order time-stepping schemes 215
5.4 Spatial discretization procedures 222
5.4.1 Galerkin formulation of the semi-discrete scheme 222
5.4.2 Galerkin formulation of θ family methods 224
5.4.3 Galerkin formulation of explicit Padé schemes 228
5.4.4 Galerkin formulation of implicit multistage schemes 229
5.4.5 Stabilization of the semi-discrete scheme 231
5.4.6 Stabilization of multistage schemes 233
5.5 Stabilized space–time formulations 241
5.6 Solved exercises 243

5.6.1 Convection–diffusion of a Gaussian hill 243
5.6.2 Transient rotating pulse 246
5.6.3 Steady rotating pulse problem 250
5.6.4 Nonlinear propagation of a step 250
5.6.5 Burgers' equation in 1D 251
5.6.6 Two-dimensional Burgers' equation 252
Appendix Least-squares in transient/relaxation problems 254

6 Viscous incompressible flows 265
6.1 Introduction 265
6.2 Basic concepts 267
6.2.1 Strain rate and spin tensors 267
6.2.2 The stress tensor in a Newtonian fluid 268
6.2.3 The Navier–Stokes equations 269
6.3 Main issues in incompressible flow problems 272
6.4 Trial solutions and weighting functions 273
6.5 Stationary Stokes problem 275
6.5.1 Formulation in terms of Cauchy stress 275
6.5.2 Formulation in terms of velocity and pressure 278
6.5.3 Galerkin formulation 279
6.5.4 Matrix problem 281
6.5.5 Solvability condition and solution procedure 283
6.5.6 The LBB compatibility condition 284
6.5.7 Some popular velocity–pressure couples 285
6.5.8 Stabilization of the Stokes problem 287
6.5.9 Penalty method 288
6.6 Steady Navier–Stokes problem 293
6.6.1 Weak form and Galerkin formulation 293
6.6.2 Matrix problem 293
6.7 Unsteady Navier–Stokes equations 294
6.7.1 Weak formulation and spatial discretization 295
6.7.2 Stabilized finite element formulation 296
6.7.3 Time discretization by fractional-step methods 297
6.8 Applications and solved exercices 306
6.8.1 Stokes flow with analytical solution 306
6.8.2 Cavity flow problem 307
6.8.3 Plane jet simulation 313
6.8.4 Natural convection in a square cavity 317