

# Applied Partial Differential Equations

with Fourier Series and  
Boundary Value Problems

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Fourth Edition



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