



Series on Knots and Everything

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QUANTUM INVARIANTS

A Study of Knots, 3-Manifolds, and Their Sets

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