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● **Mathematics, Finance and Risk**  
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# The Concepts and Practice of Mathematical Finance

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CAMBRIDGE

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