

THIRD EDITION

A First Course in the

Finite Element Method



Daryl L. Logan

Contents

1 Introduction

1

Prologue	1
1.1 Brief History	2
1.2 Introduction to Matrix Notation	3
1.3 Role of the Computer	6
1.4 General Steps of the Finite Element Method	6
1.5 Applications of the Finite Element Method	13
1.6 Advantages of the Finite Element Method	18
1.7 Computer Programs for the Finite Element Method	19
References	22
Problems	25

2 Introduction to the Stiffness (Displacement) Method

26

Introduction	26
2.1 Definition of the Stiffness Matrix	26
2.2 Derivation of the Stiffness Matrix for a Spring Element	27
2.3 Example of a Spring Assemblage	32
2.4 Assembling the Total Stiffness Matrix by Superposition (Direct Stiffness Method)	35
2.5 Boundary Conditions	37
2.6 Potential Energy Approach to Derive Spring Element Equations	50
References	58
Problems	59

3 Development of Truss Equations

63

Introduction	63
3.1 Derivation of the Stiffness Matrix for a Bar Element in Local Coordinates	63
3.2 Selecting Approximation Functions for Displacements	69
3.3 Transformation of Vectors in Two Dimensions	71
3.4 Global Stiffness Matrix	74
3.5 Computation of Stress for a Bar in the x - y Plane	78
3.6 Solution of a Plane Truss	80
3.7 Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space	87
3.8 Use of Symmetry in Structure	92
3.9 Inclined, or Skewed, Supports	95
3.10 Potential Energy Approach to Derive Bar Element Equations	101
3.11 Comparison of Finite Element Solution to Exact Solution for Bar	112
3.12 Galerkin's Residual Method and Its Application to a One-Dimensional Bar	116
References	119
Problems	120

4 Development of Beam Equations

137

Introduction	137
4.1 Beam Stiffness	138
4.2 Example of Assemblage of Beam Stiffness Matrices	143
4.3 Examples of Beam Analysis Using the Direct Stiffness Method	145
4.4 Distributed Loading	154
4.5 Comparison of the Finite Element Solution to the Exact Solution for a Beam	165
4.6 Beam Element with Nodal Hinge	171
4.7 Potential Energy Approach to Derive Beam Element Equations	176
4.8 Galerkin's Method for Deriving Beam Element Equations	179
References	181
Problems	181

Introduction	188
5.1 Two-Dimensional Arbitrarily Oriented Beam Element	188
5.2 Rigid Plane Frame Examples	192
5.3 Inclined or Skewed Supports—Frame Element	211
5.4 Grid Equations	212
5.5 Beam Element Arbitrarily Oriented in Space	229
5.6 Concept of Substructure Analysis	234
References	240
Problems	240

**6 Development of the Plane Stress
and Plane Strain Stiffness Equations****264**

Introduction	264
6.1 Basic Concepts of Plane Stress and Plane Strain	265
6.2 Derivation of the Constant-Strain Triangular Element Stiffness Matrix and Equations	270
6.3 Treatment of Body and Surface Forces	284
6.4 Explicit Expression for the Constant-Strain Triangle Stiffness Matrix	289
6.5 Finite Element Solution of a Plane Stress Problem	291
References	301
Problems	301

**7 Practical Considerations in Modeling;
Interpreting Results; and Examples of
Plane Stress/Strain Analysis****307**

Introduction	307
7.1 Finite Element Modeling	308
7.2 Equilibrium and Compatibility of Finite Element Results	318
7.3 Convergence of Solution	320
7.4 Interpretation of Stresses	321
7.5 Static Condensation	323

7.6 Flowchart for the Solution of Plane Stress/Strain Problems	327
7.7 Computer Program Results for Some Plane Stress/Strain Problems	328
References	331
Problems	332

8 Development of the Linear-Strain Triangle Equations **344**

Introduction	344
8.1 Derivation of the Linear-Strain Triangular Element Stiffness Matrix and Equations	344
8.2 Example LST Stiffness Determination	349
8.3 Comparison of Elements	352
References	354
Problems	355

9 Axisymmetric Elements **358**

Introduction	358
9.1 Derivation of the Stiffness Matrix	358
9.2 Solution of an Axisymmetric Pressure Vessel	368
9.3 Applications of Axisymmetric Elements	376
References	380
Problems	381

10 Isoparametric Formulation **386**

Introduction	386
10.1 Isoparametric Formulation of the Bar Element Stiffness Matrix	386
10.2 Rectangular Plane Stress Element	392
10.3 Isoparametric Formulation of the Plane Element Stiffness Matrix	395
10.4 Gaussian Quadrature (Numerical Integration)	404
10.5 Evaluation of the Stiffness Matrix and Stress Matrix by Gaussian Quadrature	407
10.6 Higher-Order Shape Functions	413
References	417
Problems	417

11 Three-Dimensional Stress Analysis

421

Introduction	421
11.1 Three-Dimensional Stress and Strain	421
11.2 Tetrahedral Element	423
11.3 Isoparametric Formulation	430
References	436
Problems	436

12 Plate Bending Element

441

Introduction	441
12.1 Basic Concepts of Plate Bending	441
12.2 Derivation of a Plate Bending Element Stiffness Matrix and Equations	445
12.3 Some Plate Element Numerical Comparisons	450
12.4 Computer Solution for a Plate Bending Problem	452
References	454
Problems	455

13 Heat Transfer and Mass Transport

458

Introduction	458
13.1 Derivation of the Basic Differential Equation	459
13.2 Heat Transfer with Convection	462
13.3 Typical Units; Thermal Conductivities, K ; and Heat-Transfer Coefficients, h	463
13.4 One-Dimensional Finite Element Formulation Using a Variational Method	464
13.5 Two-Dimensional Finite Element Formulation	478
13.6 Line or Point Sources	487
13.7 One-Dimensional Heat Transfer with Mass Transport	490
13.8 Finite Element Formulation of Heat Transfer with Mass Transport by Galerkin's Method	491
13.9 Flowchart and Examples of a Heat-Transfer Program	495

References	499
Problems	499

14 Fluid Flow **508**

Introduction	508
14.1 Derivation of the Basic Differential Equations	508
14.2 One-Dimensional Finite Element Formulation	513
14.3 Two-Dimensional Finite Element Formulation	521
14.4 Flowchart and Example of a Fluid-Flow Program	526
References	527
Problems	528

15 Thermal Stress **532**

Introduction	532
15.1 Formulation of the Thermal Stress Problem and Examples	532
Reference	553
Problems	554

16 Structural Dynamics and Time-Dependent Heat Transfer **559**

Introduction	559
16.1 Dynamics of a Spring-Mass System	559
16.2 Direct Derivation of the Bar Element Equations	561
16.3 Numerical Integration in Time	565
16.4 Natural Frequencies of a One-Dimensional Bar	577
16.5 Time-Dependent One-Dimensional Bar Analysis	581
16.6 Beam Element Mass Matrices and Natural Frequencies	586
16.7 Truss, Plane Frame, Plane Stress/Strain, Axisymmetric, and Solid Element Mass Matrices	591
16.8 Time-Dependent Heat Transfer	595
16.9 Computer Program Example Solutions for Structural Dynamics	602
References	609
Problems	610

Appendix A Matrix Algebra

616

Introduction	616
A.1 Definition of a Matrix	616
A.2 Matrix Operations	617
A.3 Cofactor or Adjoint Method to Determine the Inverse of a Matrix	624
A.4 Inverse of a Matrix by Row Reduction	626
References	628
Problems	628

Appendix B Methods for Solution of Simultaneous Linear Equations

630

Introduction	630
B.1 General Form of the Equations	630
B.2 Uniqueness, Nonuniqueness, and Nonexistence of Solution	631
B.3 Methods for Solving Linear Algebraic Equations	632
B.4 Banded-Symmetric Matrices, Bandwidth, Skyline, and Wavefront Methods	643
References	649
Problems	650

Appendix C Equations from Elasticity Theory

652

Introduction	652
C.1 Differential Equations of Equilibrium	652
C.2 Strain/Displacement and Compatibility Equations	654
C.3 Stress/Strain Relationships	656
Reference	659

Appendix D Equivalent Nodal Forces

660

Problems	660
----------	-----

Appendix E Principle of Virtual Work**663**

References 666

Answers to Selected Problems**667**

Index**689**